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THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 6

Inequalities

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The Open University

Mathematics Foundation Course Unit 6

INEQUALITIES

Prepared by the Mathematics Foundation Course Team

Correspondence Text 6

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Contents

Objectives

After working through this unit you should be able to:

(i) manipulate inequalities;

(ii) solve sets of simultaneous inequalities of the form

$$\left.\begin{array}{l} f(x) < 0 \\ g(x) < 0 \\ h(x) < 0, \end{array}\right\}$$

where the functions f, g and h are linear and have domains and codomains which are sets of real numbers;

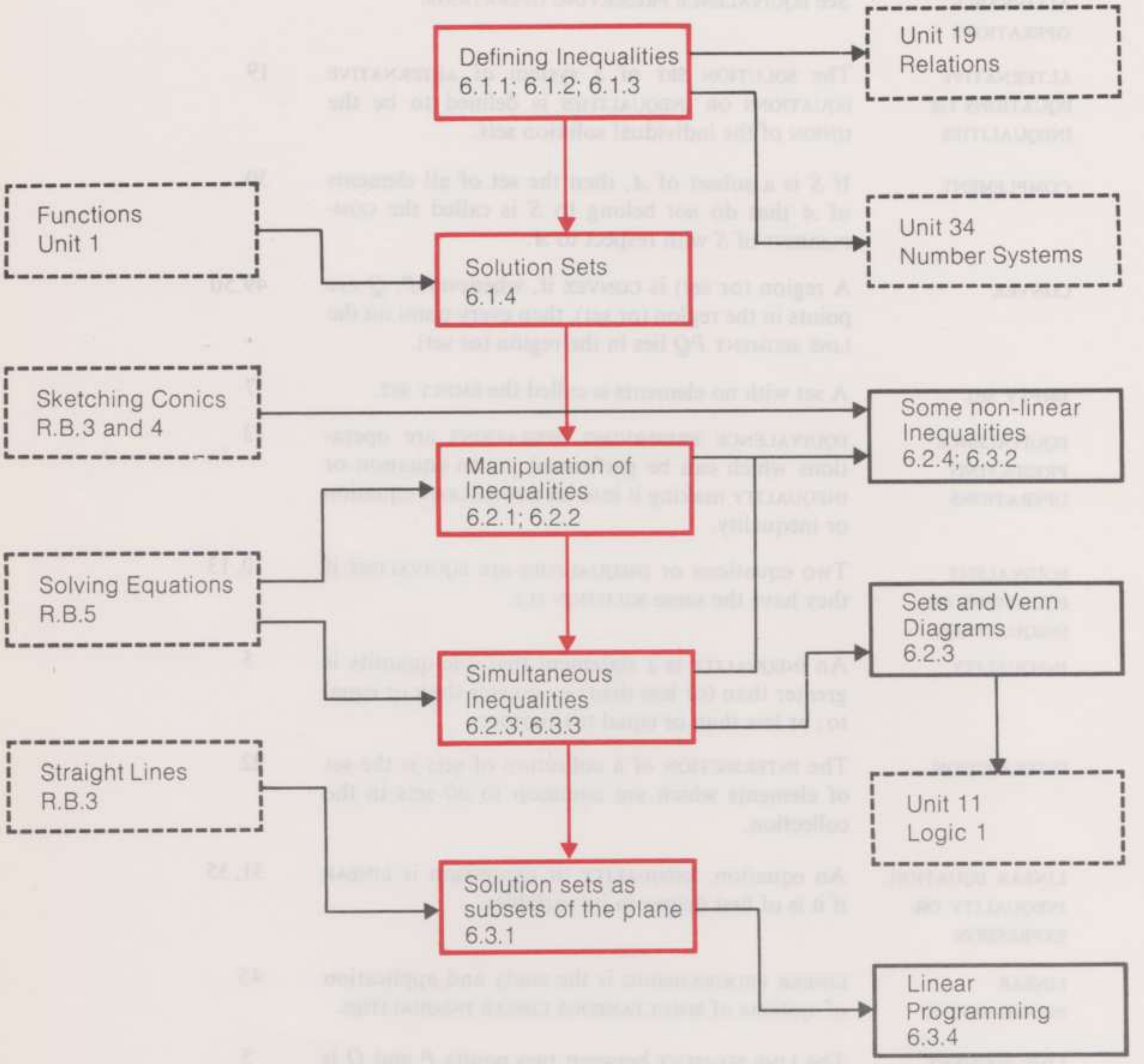
(iii) solve inequalities of the form $f(x) < 0$, where f is a polynomial function with domain and codomain which are sets of real numbers;

(iv) illustrate two-dimensional regions which represent the solution set of a system of simultaneous linear inequalities in 2 unknowns;

(v) maximize or minimize a linear expression over a region of the type in objective (iv).

N.B.
Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

Terms which are defined in this glossary are printed in CAPITALS.

		Page
ALLOWABLE OPERATIONS	See EQUIVALENCE PRESERVING OPERATIONS.	
ALTERNATIVE EQUATIONS OR INEQUALITIES	The SOLUTION SET of a system of ALTERNATIVE EQUATIONS OR INEQUALITIES is defined to be the UNION of the individual solution sets.	19
COMPLEMENT	If S is a subset of A, then the set of all elements of A that do *not* belong to S is called the COMPLEMENT of S with respect to A.	30
CONVEX	A region (or set) is CONVEX if, whenever P, Q are points in the region (or set), then every point on the LINE SEGMENT PQ lies in the region (or set).	49, 50
EMPTY SET	A set with no elements is called the EMPTY SET.	7
EQUIVALENCE PRESERVING OPERATIONS	EQUIVALENCE PRESERVING OPERATIONS are operations which can be performed on an equation or INEQUALITY making it into an EQUIVALENT equation or inequality.	13
EQUIVALENT EQUATIONS OR INEQUALITIES	Two equations or INEQUALITIES are EQUIVALENT if they have the same SOLUTION SET.	10, 13
INEQUALITY	An INEQUALITY is a statement that one quantity is greater than (or less than; or greater than or equal to; or less than or equal to) another.	5
INTERSECTION	The INTERSECTION of a collection of sets is the set of elements which are common to *all* sets in the collection.	22
LINEAR EQUATION, INEQUALITY OR EXPRESSION	An equation, INEQUALITY or expression is LINEAR if it is of first degree in its variables.	31, 35
LINEAR PROGRAMMING	LINEAR PROGRAMMING is the study and application of systems of SIMULTANEOUS LINEAR INEQUALITIES.	45
LINE SEGMENT	The LINE SEGMENT between two points P and Q is the set of points lying between P and Q on the straight line joining them.	5
NEGATIVE	A real number x is NEGATIVE if $-x$ IS POSITIVE.	4
NON-LINEAR EQUATION, INEQUALITY OR EXPRESSION.	A NON-LINEAR equation, INEQUALITY or expression is one that is not LINEAR.	31, 35
NUMBER LINE	A straight line, extending indefinitely in each direction, and used to represent the real numbers, is called a NUMBER LINE.	5
POLYHEDRAL CONVEX SET	A POLYHEDRAL CONVEX SET is a CONVEX SET in n dimensions, bounded by segments of planes in $n-1$ dimensions (i.e. points in the 1-dimensional case, line segments in the 2-dimensional case, 2-dimensional polyhedral convex sets in the 3-dimensional case, and so on).	50

		Page
POSITIVE	A real number x is POSITIVE if $x > 0$. (Alternatively, the set of positive elements of R may be taken as "given" in which case the inequalities "$<$" and "$>$" are defined by: $a < b$ if $b - a$ is positive; $a > b$ if $a - b$ is positive.)	4
SIMULTANEOUS EQUATIONS OR INEQUALITIES	The SOLUTION SET of a system of SIMULTANEOUS equations or INEQUALITIES is defined to be the INTERSECTION of the individual solution sets.	19
SOLUTION SET	The SOLUTION SET of an equation, INEQUALITY or system of such, is the set of elements (of R, $R \times R$, etc.) for which all the statements of the system are true.	7
SOLVE AN EQUATION INEQUALITY OR SYSTEM OF THESE	To SOLVE an equation, INEQUALITY or system of these is to find its SOLUTION SET.	7
TRANSITIVE PROPERTY	Equalities and INEQUALITIES have the TRANSITIVE PROPERTY — that is to say, if $a\,T\,b$ and $b\,T\,c$, then $a\,T\,c$ where T is $=$, $<$, $>$, $\leqslant$, or $\geqslant$.	11
UNION	The UNION of a collection of sets is the set of all elements belonging to *at least one* of the sets in the collection.	23
UNIQUE	The statement: "There is a UNIQUE x with a given property" means that there is exactly *one* element x with that property.	2

Notation

The symbols are presented in the order in which they appear in the text.

		Page
$f(x)$	The image of the element x under the function f.	x
R	The set of real numbers.	x
$R \times R$	The Cartesian product of R and R.	x
$a = b$	a is equal to b (or: $a - b$ is zero).	x
$a < b$	$b - a$ is positive (see Glossary).	5
$a > b$	$a - b$ is positive (see Glossary).	5
$a \leqslant b$ or $a \not> b$	$a - b$ is not positive (see Glossary).	5
$a \geqslant b$ or $a \not< b$	$b - a$ is not positive (see Glossary).	5
$\in$	"is an element of".	5
$\{x : x$ has property $P\}$	The set of all x such that x has property P.	7
$\varnothing$	The empty set (i.e. the set containing no elements).	7
$\{a, b, \ldots\}$	The set containing the elements $a, b, \ldots$	7
$[a, b]$	The set of real numbers between (and including) a and b; that is, $\{x : x \in R, a \leqslant x \leqslant b\}$.	15
$]a, b[$	The set of real numbers between (but excluding) a and b; that is, $\{x : x \in R, a < x < b\}$.	15
$[a, b[$	$\{x : x \in R, a \leqslant x < b\}$.	15
$]a, b]$	$\{x : x \in R, a < x \leqslant b\}$.	15
$A \cap B$	The intersection of sets A and B (see Glossary).	22
$A \cup B$	The union of sets A and B (see Glossary).	23
A'	The complement of A with respect to some set, given by the context (see Glossary).	30
(a, b)	The ordered pair of elements a and b.	34

Bibliography

E. Beckenbach and R. Bellman, *An Introduction to Inequalities* (Random House, 1961).
This is an easy-to-read book which covers most of the material of this unit. It contains interesting chapters on maximization and minimization problems, and the idea of a metric is introduced for those interested in further reading.

C. B. Allendoerfer and C. O. Oakley, *Principles of Mathematics*, 2nd ed. (McGraw-Hill, 1963).
Equations and Inequalities are discussed in Chapter 5.

N. D. Kazarinoff, *Analytic Inequalities* (Holt, Rinehart and Winston, 1964).
This is a more difficult book, but it is thoroughly recommended for anybody who has doubts as to the importance of the subject. In 85 pages the book moves from the definition of inequalities and a discussion of how to manipulate them, through a nice chapter on geometric inequalities, to a long list of problems, some of which are still unsolved.

6.0 INTRODUCTION

6.0

Introduction

You may have the impression that mathematics deals with exact, closed problems like finding the area of a given triangle, or calculating how long it would take six men to dig a trench of given dimensions. We tried to dispel this impression in *Unit 2, Errors and Accuracy*, where we showed you that even if a problem asks for an exact answer, it is not always possible to find one.

But there is another class of problem which is of considerable concern to mathematicians. These problems demand not a number for an answer, but a description of a situation. They often arise when one is trying to maximize or minimize a quantity. For example, we may be given the length of the perimeter of a triangle, and asked which triangle with such a perimeter has maximum area. Or, instead of being asked how long it takes six men to dig a trench, we might be told that two men can work at a certain rate for two hours and then slack off, a third is very good at excavating earth from the bottom of the trench, two others are good with a wheelbarrow, and the sixth makes the best cup of tea on the site. How should we deploy the men to get the work done as quickly as possible?

A legendary example of such a problem is known in the mathematical literature as "Dido's problem". Dido was a Phoenician princess, and when seeking land for a new settlement in North Africa, she obtained a not over-generous concession from a local chief to occupy as much land as she could encompass with a cowhide. But Dido was not a princess for nothing: being rather clever, she cut the skin into very thin strips, tied them together and enclosed a valuable piece of land on which was built the city of Carthage. In mathematical terms, Dido was faced with the problem of deciding which closed curve of given length of perimeter encloses the maximum area.

Newton tackled a similar problem, which is highly relevant today. A body (a rocket, say, or a bullet) moving through the air meets a certain resistance. What shape should the body be to minimize this resistance?

All these problems of satisfying certain constraints involve the idea of inequality. By an inequality, we mean a statement that one quantity is greater than another, or that it is less than another. Of all boundaries of given length L, a circular boundary encloses the largest area (which is $L^2/4\pi$), and so every other boundary encloses an area *less than* $L^2/4\pi$. In Newton's problem, all body shapes except the optimum offer a resistance *greater than* the optimum.

A feature of all these problems is that we use the information available to select from a wide range of possibilities one set of situations which satisfy all the constraints which have been imposed. There may be just one such situation, as in Dido's problem, where the required situation was a circle, or there may be a whole class of situations. For example, the legal constraint that the depth of tread on the tyre of a motor car must be greater than 1 millimetre restricts the possible depths of tread not to just one thickness but to a whole class of thicknesses, i.e. all those greater than 1 millimetre. Again, a pathologist can give a rough estimate of a time of death if rigor mortis is present in the body. He may know that, under the conditions prevailing at the time, rigor mortis sets in about 24 hours after death and disappears 24 hours later. So he may be able to say that death occurred *more than* a day ago but *less than* two days ago.

The subject of inequalities is the topic under discussion in this unit. An understanding of inequalities is necessary for tackling problems of maximization and minimization such as those that we have been discussing. But their use is more widespread than this. Soon we shall be embarking on the first of our units on calculus, and throughout calculus an

understanding of inequalities, and an ability to manipulate them, is essential.

All the inequalities which we shall discuss in this unit are concerned with real numbers. You have a working knowledge of numbers which is sufficient to perform the techniques we shall be discussing, but, so that you understand the techniques, and so that you know just which properties of the real numbers we are using, we start this unit by discussing some of the properties of the real numbers and what we mean by the word "inequality".

We shall also discuss the idea of the solution set of an inequality containing one or more unknowns. The solution set of an *equation*, such as $f(x) = 0$, involving an unknown x, is the set of all elements in the domain of the function f which map to zero under f. The solution set of an *inequality*, such as $f(x) > 0$, is the set of all elements in the domain of f which have images which are greater than zero under f. So the techniques required for the solution of such an inequality depend on the function f. One way to approach the problem is to consider particular classes of function in turn. Thus, after making sure of the techniques for manipulating inequalities, we shall consider inequalities $f(x) > 0$ in which f is a mapping from R to R: first of all linear functions (i.e. those whose graphs are straight lines) and then some other types. We then deal with some inequalities $f(x, y) > 0$ in which f is a mapping from $R \times R$ to R. We apply some of these concepts by introducing you to an application of inequalities, known as linear programming, which is used in management studies for problems involving the most profitable allocation of resources. The unit ends with a discussion of convex sets, which leads to some simple and elegant mathematics.

6.1 WHAT IS AN INEQUALITY?

6.1

6.1.1 The Real Number System

6.1.1

Discussion

In *Unit 2, Errors and Accuracy*, and in *Unit 4, Finite Differences*, we have been talking a lot about how to calculate in the set of real numbers. We are so familiar with the real number system that we take many of its properties for granted. In this section we shall discuss the fundamental properties of this system, some of which have been mentioned in *Unit 3, Operations and Morphisms*.

It is important, for two main reasons, to single out those properties of the real numbers which can be regarded as fundamental:

(i) to form a basis for proofs: it is important in any proof to state clearly the assumptions that are being made; and

(ii) so that we can recognize other systems which have the same properties, and distinguish them from systems with different properties.

The second reason may not mean very much to you yet, but we saw in *Unit 3, Operations and Morphisms*, how ideas like those of commutativity and associativity, with which we are familiar in the real number system, are used in the discussion of operations which have nothing to do with numbers. In the same way, by listing a set of basic properties of the real numbers, from which we can deduce all the other properties of real numbers, we can characterize their structure. Then it may happen that we come across another set with operations defined on it having precisely these basic properties, and so all the results with which we are familiar for real numbers will carry straight over to this new system. Indeed, we shall meet examples later in this course of other systems which possess some or all of the properties of real numbers, and where precisely this sort of exercise can be carried out. We shall be discussing the real numbers in much more detail in *Unit 34, Number Systems*. In the present unit we shall discuss those properties of the real number system which are particularly relevant to the problem in hand.

Main Text

The following list of four properties (which we label as Re (1) to Re (4) for reference) can be taken as a starting point.

Some Properties of the Real Numbers

Re (1) The set of real numbers is closed under the binary operations of addition and multiplication. That is, for any real numbers a and b, both the sum $a + b$ and the product $a \cdot b$ are uniquely defined and are themselves real numbers. Re (1)

Re (2) Addition and multiplication are both commutative and associative, and multiplication is distributive over addition (see *Unit 3, Operations and Morphisms*). That is, Re (2)

$$\left.\begin{aligned} a + b &= b + a \\ a \cdot b &= b \cdot a \end{aligned}\right\} \textit{commutative property}$$

$$\left.\begin{aligned} a + (b + c) &= (a + b) + c \\ a \cdot (b \cdot c) &= (a \cdot b) \cdot c \end{aligned}\right\} \textit{associative property}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \textit{distributive property}$$

where a, b and c are any real numbers.

Re (3) There are two real numbers, 0 and 1 (which are not equal), with the properties that, for any real number a, Re (3)

$$a + 0 = a \qquad \text{and} \qquad a \cdot 1 = a.$$

Re (4) For any real numbers a and b, the equation **Re (4)**

$$a + x = b$$

has a unique real solution, written $x = b - a$. (When $b = 0$, the notation $0 - a$ is abbreviated to $-a$.) For any real numbers a and b, where a is not 0, the equation

$$a \cdot x = b$$

has a unique real solution, written b/a or $\dfrac{b}{a}$. (By unique, we mean that there is just *one* real number which has the stated property.) **Definition 1** **Supplementary Material**

Example 1 **Example 1**

If you are interested to see how some other properties of the real numbers can be deduced from these four, you may like to consider the following five properties, and the way in which they are deduced. The important thing in each case is that every manipulation is justified from the above list of properties. Manipulations whose justification lies outside the list can be used only if they have themselves been justified from the list; for example, when we have established property Re (I) below, we use it in establishing property Re (II). Don't worry if you find this rather difficult. You are advised to *read* the rest of this section but, if it does not appeal to you, continue with Section 6.1.2.

Re (I) *The element* 0 *of property* Re (3) *is unique.* **Re (I)**

Deduction (I)

Let c be any real number such that $a + c = a$ for every real number a. (By Re (3), there is at least one real number with this property.) The equation

$$a + c = a$$

is of the same form as the equation

$$a + x = b$$

of Re (4), with c substituted for x, and a for b. By Re (4), the number c satisfying this equation is unique: that is,

$$c = 0$$

To put it slightly differently: *the number* $a - a$ *is* 0, *for every real number* a.

Re (II) *The product* $b \cdot 0$ *is* 0, *for every real number* b. **Re (II)**

Deduction (II)

Let a be any real number. By Re (3)

$$a + 0 = a$$

Now let b be any real number. It follows that:

$$b \cdot (a + 0) = b \cdot a$$

and from the distributive property in Re (2), we have

$$b \cdot a + b \cdot 0 = b \cdot a$$

By Re (4), this can be regarded as an equation for $b \cdot 0$, having the unique solution

$$b \cdot 0 = b \cdot a - b \cdot a$$

$$= 0, \qquad \text{by Re (I)}$$

Re (III) *The element 1 of property* Re (3) *is unique.* Re (III)

Deduction (III)

Let c be any real number such that $a \cdot c = a$ for every real number a. (By Re (3), there is at least one real number with this property.) The equation

$$a \cdot c = a$$

is of the same form as the equation

$$a \cdot x = b$$

of Re (4), with c substituted for x, a for b, and a restricted to be not equal to 0. By Re (4), the number c satisfying this equation is unique; that is,

$$c = 1$$

To put it slightly differently: *the number a/a is 1, for every real number a such that a is not 0.*

Re (IV) *The product $(-a) \cdot (-b)$ is $a \cdot b$, for all real numbers a and b.* Re (IV)

Deduction (IV)

This follows by considering the expression

$$a \cdot b + a \cdot (-b) + (-a) \cdot (-b).$$

(Notice that we do not need any more brackets, by the associativity of + in Re (2).)

The sum of the first two terms is

$$a \cdot (b + (-b))$$

by Re (2).

But $b + (-b)$ is 0 by the definition of "$-b$" (see Re (4)), and $a \cdot 0$ is 0 by Re (II). So the sum of the first two terms is 0, and the whole expression adds up to $(-a) \cdot (-b)$ (see Re (3)).

Similarly, the sum of the last two terms is 0, and the whole expression adds up to $a \cdot b$. But, as we have already remarked, + is associative (Re (2)), so the two ways of simplifying the whole expression must lead to the same result.

Thus (since by Re (1) the total sum must be unique),

$$a \cdot b = (-a) \cdot (-b)$$

Re (V) *The product $a \cdot (-b)$ is $-(a \cdot b)$, for all real numbers a and b.* Re (V)

Deduction (V)

This follows directly from the fact (derived in the course of Deduction (IV)) that

$$a \cdot b + a \cdot (-b) = 0$$

■

6.1.2 More Properties of the Real Number System

6.1.2

Main Text

As we have mentioned, some of the properties listed in Re (1) to Re (4) have already been discussed in *Unit 3*. We shall refer to them often in this course. All the properties will play an important part in our discussion of *Groups* (*Units 30 and 33*).

But there are two other sets of properties of a slightly different nature which we also take for granted. We assume, when developing proofs, what we might call three rules of logic, or three properties of the sign "=" (which we label as Eq (1) to Eq (3) for reference):

Eq (1) For any real number a, we have $a = a$. **Eq (1)**

Eq (2) If $a = b$, then we have $b = a$. **Eq (2)**

Eq (3) If $a = b$ and $b = c$, then we have $a = c$. **Eq (3)**

These three properties enable us to carry out the manipulations required when proving theorems. Again, they may seem to you to be so basic that they are too simple to be interesting. But when you read *Unit 19, Relations*, you will see that there is far more in them than meets the eye. In particular, in view of the fact that no two different things can be precisely the same, what do we really mean by, for example, "$a = b$"?

The third set of properties of the real numbers to which we want to draw your attention expresses the fact that they are *ordered*. For example, the numbers

$$-1, 2, -0.5, 7, 4.3$$

can be arranged in order,

$$-1, -0.5, 2, 4.3, 7$$

so that if the number b stands to the right of the number a in this ordered list, then $(b - a)$ is positive. Notice that the last clause expresses this property in terms of positive numbers. Can we express this idea of order entirely in terms of positive numbers? It is a remarkable fact that we can do so as follows, and that the theory of inequalities stems from these four properties (which we label as Ord (1) to Ord (4) for reference):

Ord (1) The set of real numbers contains a subset of positive real numbers, which has the following properties: **Ord (1)** **Definition 1**

Ord (2) The sum of two positive real numbers is positive. **Ord (2)**

Ord (3) The product of two positive real numbers is positive. **Ord (3)**

Ord (4) Given any real number a, $a \neq 0$, *either* a is positive *or* $-a$ is positive (but not both). **Ord (4)**

Definition **Definition 2**

If the real number a is positive, then the number $-a$ is called *negative*.

(You may be wondering whether these ordering properties can be deduced from the four properties Re (1) to Re (4) which were listed on page 1. In fact they cannot, and when we discuss *Complex Numbers* (*Units 27 and 29*), you will meet a system which has properties Re (1) to Re (4) but *not* the properties Ord (1) to Ord (4).)

These ordering properties enable us to define the symbols which are the subject of this unit.

Exercise 1 (do not worry if you find this difficult)

Exercise 1
(5 minutes)

Use Re (IV) of Example 6.1.1.1, together with Ord (3), to show that the product of two negative numbers is a positive number. Similarly, use Re (V), together with Ord (3), to show that the product of a positive number and a negative number is a negative number.

Deduce from this latter result that the number 1 of Re (3) is positive. ■

6.1.3 Inequalities: Some Notation and Definitions

6.1.3

Notation

Notation ***

The formulas "$a < b$" (read "a is less than b") and "$b > a$" (read "b is greater than a") both mean that the number $b - a$ is positive. Notation 1 **

The formula "$a \leqslant b$" (or "$b \geqslant a$") means that a is either less than or equal to b, that is, $b - a$ is positive or zero. Notation 2 **

Formulas such as "$a < b$", "$b \geqslant a$" are called inequalities. Definition 1 ***

An alternative formula to "$a \leqslant b$" is "$a \ngtr b$" (read "a is not greater than b"), and similarly "$a \geqslant b$" is sometimes written "$a \nless b$" (read "a is not less than b"). Notation 3 **

Notice that $a > 0$ means that $a - 0 = a$ is positive; so "$a > 0$" is an abbreviation for "a is positive".

Exercise 1

Exercise 1 (2 minutes)

Which of the following statements are true and which are false?

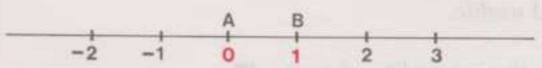
(i)	$6 > 3$	TRUE/FALSE
(ii)	$3 > 3$	TRUE/FALSE
(iii)	$3 < 3$	TRUE/FALSE
(iv)	$3 \leqslant 3$	TRUE/FALSE
(v)	$3 \geqslant 3$	TRUE/FALSE
(vi)	$\frac{1}{2} < -10$	TRUE/FALSE
(vii)	$\frac{1}{8} > \frac{1}{2}$	TRUE/FALSE
(viii)	$\frac{19}{20} < \frac{20}{21}$	TRUE/FALSE
(ix)	$-10 < -8$	TRUE/FALSE
(x)	$x^2 \leqslant 0$, for all $x \in R$.	TRUE/FALSE

■

The Number Line

Discussion **

The ordering properties are usefully illustrated as follows. Take a straight line and select on it any two points A and B. Label the left hand point 0 and the right hand point 1, and take the distance between these two points as a standard unit of length.



Points on the line corresponding to the other integers are located by marking standard units of length along the line in both directions: positive numbers are then arranged in order to the right of 0 and negative numbers to the left. The points representing numbers like $\frac{m}{n}$, where m and n are non-zero integers and $m < n$, can be located by dividing the line segment between 0 and 1 into n equal segments. The location of all the rational numbers follows. A proof that the irrational numbers, like $\sqrt{2}$, are represented uniquely on this number line is beyond our scope. You may wonder why we make so much fuss — after all, we have all seen rulers—but the *proof* that there is a one-one mapping from the set of real numbers to the set of points on a line involves philosophical difficulties and is no easy matter.

See Glossary

Definition 2 *

The statement "$a < b$" has the interpretation that the point representing a on the number line lies to the left of the point representing b.

Solution 6.1.2.1

Solution 6.1.2.1

By definition, a real number c is negative if $c = -a$ for some positive real number a. Let c and d be any two negative numbers:

$$c = -a, \qquad d = -b,$$

where a and b are positive. Then, by Re (IV),

$$c \cdot d = (-a) \cdot (-b)$$
$$= a \cdot b$$

So, by Ord (3), $c \cdot d$ is positive.

Similarly, Using Re (V) and the above symbols:

$$a \cdot d = a \cdot (-b)$$
$$= -(a \cdot b)$$

So, by Ord (3), $a \cdot (-b)$ is negative.

To deduce that 1 is positive, we notice by Re (3) that $1 \cdot a = a$ for all real numbers a. Suppose that 1 is negative and choose a positive number for a. Then by the latter result $1 \cdot a$ is negative, and therefore cannot be the same as a which is positive. Thus our supposition that 1 is negative is false. Since 1 is not 0 (compare Re (3) and Re (II)), it must be positive (Ord (4)). ■

Solution 1

Solution 1

(i)	TRUE	(ii)	FALSE	(iii)	FALSE
(iv)	TRUE	(v)	TRUE	(vi)	FALSE
(vii)	FALSE				

(viii) TRUE. For this one, the best way is to go back to the definition of "$<$". That is, work out $\frac{20}{21} - \frac{19}{20}$ and see whether this number is indeed positive, in which case the answer to (viii) is "true". Thus:

$$\frac{20}{21} - \frac{19}{20} = \frac{20 \cdot 20 - 19 \cdot 21}{21 \cdot 20}$$
$$= \frac{400 - 399}{21 \cdot 20}$$

which is positive. This demonstrates the importance of having a definition which is both *precise* and *usable.*

(ix) TRUE

(x) FALSE. Any real number, x, satisfies the inequality $x^2 \geqslant 0$. ■

6.1.4 Solution Sets of Equations and Inequalities

6.1.4

Definitions

Most of the remainder of this unit will be concerned with techniques for "solving inequalities". What do we mean by this? You may be more familiar with the idea of "solving" an equation. What do we mean when we say

"Solve the equation $f(x) = 0$"?

Let f be a mapping with domain, A, and codomain, B, which are sets of real numbers.
The phrase solve the equation $f(x) = 0$ means

find the set of all elements of A which map to 0 under f

We can write this set as

$\{x : x \in A, f(x) = 0\}$

and we call it the solution set of the equation $f(x) = 0$. Note that this set may not have any members, in which case we call it the empty set and denote it by $\varnothing$. Thus

Definition 1

Definition 2

$\{x : x \in R, x^2 + 1 = 0\} = \varnothing$

Similarly, solve the inequality $f(x) > 10$, for example, means find the set of all elements of A which map to numbers greater than 10 under f.

Definition 3

We can write this set as

$\{x : x \in A, f(x) > 10\}$

and we call it the solution set of the inequality $f(x) > 10$

Definition 4

Exercise 1

Exercise 1
(3 minutes)

Illustrate on a number line the following sets. For example, the answer to (i) is

(i) $\{1, 1.5, 3.75\}$
(ii) $\{x : x \in R, 3 + x = 2\}$
(iii) $\{x : x \in R, x^2 + 3x + 2 = 0\}$
(iv) $\{x : x \in R, x > 0\}$
(v) $\{x : x \in R, x \leqslant 2\}$

■

Exercise 2

Exercise 2
(5 minutes)

(i) Find a positive integer N such that

$$\frac{1}{N^2 + N + 1} < \frac{1}{10}$$

(ii) Find a positive integer N such that

$$\frac{1}{n+1} < 0.01 \text{ for ALL integers } n \text{ greater than } N$$

That is, find, by determining a value for N, a set $A = \{n : n > N\}$ such that all elements of A belong to the solution set of the inequality (or, more briefly, such that A is a subset of the solution set).
(iii) Find a real positive number δ such that

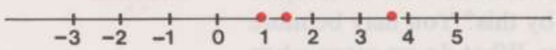
$$\delta \sin \delta < 0.1$$

(δ is the Greek letter called "delta".)

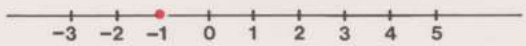
■

Solution 1 **Solution 1**

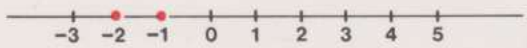
(i)



(ii)

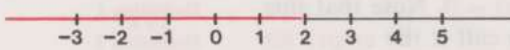


(iii) $\{x : x \in R,\, x^2 + 3x + 2 = 0\} = \{-1, -2\}$

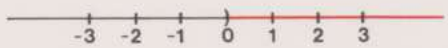


(iv)

(v)



In cases (iv) and (v) it is not clear from the illustrations whether or not the respective end-points, 0 and 2, are included in the solution sets. There is a way of indicating this which is often used. To indicate that 0 is excluded in case (iv) we draw the following diagram:



To indicate that 2 is included in case (v) we draw:

Similarly, the diagram:

represents the set $\{x : x \in R,\, 1 \leqslant x < 6\}$ in which 1 is included and 6 is excluded. ■

We shall not be using illustrations such as these unless we wish to emphasize the inclusion or exclusion of an end-point.

Solution 2 **Solution 2**

(i) $N = 3$ will serve the purpose. One can find this number, and many others, by trial and error.

(ii) As n gets larger and larger, $\dfrac{1}{n+1}$ gets smaller and smaller. If we can find *any* one value of N such that $\dfrac{1}{N+1} < 0.01$, then we know that $\dfrac{1}{n+1} < 0.01$ for all values of n greater than N. One such value for N is

$$N = 100$$

(iii) $\delta = 0.3$ is a possibility. ■

This is an important exercise, in that it leads up to the ideas of approximation and limits, topics which are dealt with more fully in *Unit 7, Sequences and Limits I.*

6.1.5 Summary

6.1.5

Summary

Since the sets which we shall discuss in this unit are sets of real numbers or pairs of real numbers, we have started by discussing some properties of the real numbers.

We have distinguished between two types of properties: firstly, the "arithmetic" properties Re (1) to Re (4), which we listed on page 1, and then the "ordering" properties Ord (1) to Ord (4), listed on page 4. The ordering properties enable us to define the term "inequality" which is the subject of this unit. We ended the section by discussing what we mean by the phrase "solve an inequality", and saw that the problem has features in common with that of solving an equation. The techniques for solving equations will be very much in our minds as we try to develop an armoury for tackling inequalities, and it was for this reason that the properties of the sign "=", which are usually taken for granted, were listed on page 4. In the next section we shall be asking how much technique we can carry over from the solution of equations to the solution of inequalities.

Note one last point. An inequality such as

$$f(x) < g(x)$$

is a statement about elements in the codomains of two functions f and g. The solution set of the inequality is the set of elements in the domains of f and g for which the statement is true. For the inequality to be meaningful we require that:

(i) the domains of f and g have elements in common;

(ii) the codomains of f and g are subsets of a set for which inequalities make sense (i.e. a set which is ordered);

but these conditions do not guarantee the existence of a non-empty solution set.

THROUGHOUT THIS UNIT, ALL INEQUALITIES WILL BE STATEMENTS ABOUT REAL NUMBERS, AND THE SOLUTION SETS WILL ALWAYS BE SETS OF REAL NUMBERS OR PAIRS OF REAL NUMBERS.

Note

Notation

With the above understanding, in future we shall often write:

$$\{x: f(x) > 0\} \quad \text{instead of} \quad \{x: x \in R, f(x) > 0\}$$

and, for example,

$$ax^2 + bx + c \geqslant 0 \quad \text{instead of} \quad ax^2 + bx + c \geqslant 0, \quad a, b, c \in R.$$

6.2 SOME TECHNIQUES FOR SOLVING INEQUALITIES

6.2

6.2.1 Manipulation of Inequalities

6.2.1

Discussion

In this section we shall discuss the operations which we may perform on each side of an inequality if we wish to leave the solution set unaltered. The purpose of this is to acquire some techniques which we can apply so that we can reduce an inequality, such as $x^2 + 6x - 4 < 0$, to a simpler form for which we can easily find the solution set. This procedure is similar to that which we apply to equations: for example, we can reduce the set $\{x : x \in R, x^2 + 3x + 2 = 0\}$ to $\{-2, -1\}$ by solving the quadratic equation.

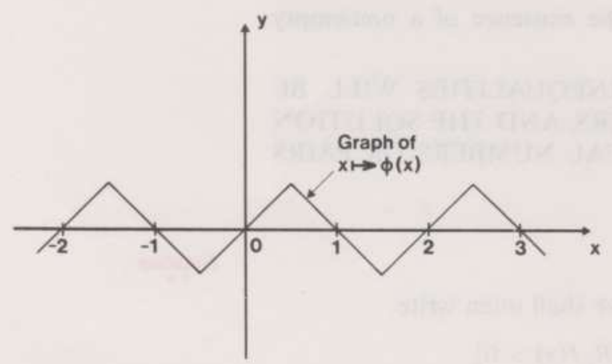
You are probably acquainted with the way in which we tackle a problem such as "solve the equation $ax + b = c$". We can solve the equation because we know that we can add or subtract numbers to or from each side of the equation; we can also multiply or divide each side by any non-zero number, and the equation still holds. Essentially, at each stage of the solution we replace an equation by another simpler equation which has the same solution set. This leads us to define the term "equivalent equations".

Definition 1

Two equations are equivalent if they have the same solution set.

A Word of Warning

Although we have several operations which convert an equation to an equivalent equation, this does not imply that any two equivalent equations can be manipulated, one to another. For example, if ϕ* is the function with domain R, and graph as shown,



then the equations

$$\phi(x) = 0$$

$$\sin \pi x = 0$$

each have solution set $\{0, \pm 1, \pm 2, \ldots,\}$ and are therefore equivalent; but this is *not* to say that $\phi(x) = 0$ can be manipulated to $\sin \pi x = 0$ by our "equivalence preserving" operations.

Exercise 1

Exercise 1
(3 minutes)

In each case state whether or not the following pairs of equations are equivalent. In each case, consider the equations as arising from mappings with domain R.

(i) $x + 1 = 4$, $\quad x = 3$
(ii) $x + 1 = 4$, $\quad x^2 + x = 4x$
(iii) $x + 1 = 4$, $\quad (x - 3)(x + 1) = 4(x - 3)$
(iv) $x + 1 = 4$, $\quad (x - 1)(x + 1) = 4(x - 1)$
(v) $x(x - 1) = 0$, $\quad x(x - 1)(x - 2) = 0$

■

* ϕ is the Greek letter called "phi".

Discussion

We have already mentioned the three properties, Eq (1) to Eq (3), of the "equality" symbol. We shall see, first of all, whether these properties are shared by inequalities. Our next line of attack will be to see whether the manipulations which change equations to equivalent equations can be carried over to inequalities. If some or all of them cannot be carried over, we shall try to deduce another set of manipulations which are applicable.

Exercise 2

Exercise 2
(5 minutes)

Using the definitions of the inequality symbols on page 5, and the order properties Ord (1) to Ord (4) on page 4, show that if the equality symbol in the three properties Eq (1) to Eq (3) on page 4 is replaced by $<$, then only one of the three properties holds. What happens if we replace $=$ by $\leqslant$? ■

This last exercise shows that the inequality relation has the so-called transitive property:

Definition 2

if $a < b$ and $b < c$, then $a < c$.

We now ask the questions: "Which of the other manipulative procedures which we use to solve equations can we carry over to inequalities?" "Can we add numbers to each side of an inequality?" "Can we multiply each side of an inequality by the same number?" etc.

Exercise 3

Exercise 3
(5 minutes)

Which of the following statements are true and which are false? (If you have any difficulty in deciding, try substituting a few numbers.) All the letters stand for real numbers.

(i) If $a < b$ then $a + c < b + c$, where c is any real number.	TRUE/FALSE
(ii) If $a < b$ then $-a < -b$.	TRUE/FALSE
(iii) If $a < b$ then $b > a$.	TRUE/FALSE
(iv) If $a < b$ then $ka < kb$ where k is any positive real number.	TRUE/FALSE
(v) If $a < b$ then $ka < kb$ where k is any real number.	TRUE/FALSE
(vi) If $a < b$ then $\frac{1}{a} < \frac{1}{b}$.	TRUE/FALSE

■

Main Text

This exercise leads us to list the following properties (which we label as Ineq (1) to Ineq (4) for reference):

Ineq (1) If $a < b$, then $a + c < b + c$, where c is any real number.

Ineq (2) If $a < b$ and $k > 0$, then $ka < kb$.

Ineq (3) If $a < b$ and $a > 0$, then $\frac{1}{a} > \frac{1}{b}$.

Ineq (4) If $a < b$ and $k < 0$, then $ka > kb$.

Supplementary Material

These properties can be deduced quite simply from the order property Ord (3) and the definitions on page 5. If you are not interested in such deductions, proceed to Exercise 4.

Deduction (1)

If $a < b$, then $b - a$ is a positive real number. But

$$(b + c) - (a + c) = b - a$$

and so $(b + c) - (a + c)$ is a positive real number. Therefore

$$a + c < b + c.$$

(continued on page 12)

Solution 1

Solution 1

(i) They are equivalent.
(ii) Not equivalent. $\{x : x + 1 = 4\} = \{3\}$,
$\{x : x^2 + x = 4x\} = \{0, 3\}$.
(iii) Equivalent. $\{x : x + 1 = 4\} = \{3\}$,
$\{x : (x - 3)(x + 1) = 4(x - 3)\} = \{3\}$.

Care must be taken in cases such as this. The solution set of

$$(x - a)(x + 1) = 4(x - a)$$

is $\{a, 3\}$. In the special case when $a = 3$, this solution set is the same as that of $x + 1 = 4$, but in some situations, we would say that the solution of the equation is "a double root at $x = 3$".

(iv) Not equivalent. This case looks similar to (iii), but multiplication by $(x - 1)$ introduces a *new* root, 1. In (iii), multiplication by $x - 3$ just gives a repetition of the root 3.
(v) Not equivalent. $\{x : x(x - 1) = 0\} = \{0, 1\}$
$\{x : x(x - 1)(x - 2) = 0\} = \{0, 1, 2\}$ ■

Solution 2

Solution 2

(i) Since $a - a = 0$, $a - a$ is not positive, and so $a \nless a$.
(ii) If $a < b$, $b - a$ is positive, and so $a - b$ is not positive, and so $b \nless a$.
(iii) If $a < b$ and $b < c$, then $b - a$ and $c - b$ are both positive. Therefore, $(b - a) + (c - b)$ is positive, but since $(b - a) + (c - b) = c - a$, this means that $c - a$ is positive and $a < c$.

For $\leqslant$, properties (i) and (iii) hold. ■

Solution 3

Solution 3

(i) TRUE.
(ii) FALSE. $3 < 5$ but $-3 > -5$
(iii) TRUE.
(iv) TRUE.
(v) FALSE. $3 < 4$ but $(-7)\cdot 3 > (-7)\cdot 4$
(vi) FALSE. $-6 < -2$ but $-\frac{1}{6} > -\frac{1}{2}$ ■

(continued from page 11)

Deduction (2)

Deduction (2)

If $a < b$, then $b - a$ is positive, and so

$$kb - ka = k(b - a)$$

is positive by Ord (3), since k is positive. Therefore,

$$ka < kb.$$

Deduction (3)

Deduction (3)

Choose $k = \dfrac{1}{a \cdot b}$ in Deduction (2). We can do this because $a \cdot b$ is positive by Ord (3), and so not equal to 0, so that $\dfrac{1}{a \cdot b}$ is defined. All we have to do is to show that k is positive, and then Deduction (2) applies here word for word.

By definition, see Re (4), page 2, k satisfies the equation

$$(a \cdot b) \cdot k = 1$$

Re (V), page 3, states that $a \cdot (-b) = -(a \cdot b)$ for all real numbers a and b. If we assume a and b are positive, then this states that a positive number times a negative number is negative. So if k is negative, 1 must

be negative. But we have already shown in Exercise 6.1.2.1 that 1 is positive, and therefore, k is not negative. k is also not zero (the assumption that it is contradicts Re (II), page 2, i.e. that $a \cdot 0 = 0$, for all real numbers a). Therefore k must be positive.

Deduction (4)

Deduction (4)

Using the result of Exercise 6.1.2.1, since k is negative and $a - b$ is negative,

$$ka - kb = k(a - b)$$

is positive, so that

$$ka > kb.$$

Exercise 4

Exercise 4
(3 minutes)

Write down the properties corresponding to the list Ineq (1)–Ineq (4) on page 11 with inequalities such as $a \geqslant b$, $a \leqslant b$, $a > b$. ■

Exercise 5

Exercise 5
(3 minutes)

Extend Ineq (3) on page 11 to the case when $a < 0$. (Be sure to consider the two cases $b < 0$, $b > 0$.) What happens if $a = 0$ or $b = 0$? ■

Discussion

Earlier we defined the concept of equivalent equations. The properties which we use to solve equations are those which preserve equivalence, i.e. leave the solution set unchanged. Similarly with inequalities, we define equivalent inequalities as inequalities with the same solution sets, and we solve inequalities by transforming one to the other using the "allowable operations" or "equivalence preserving operations" which we have developed in this section. For example, since we are allowed to add a number (by Ineq (1)) to each side of an inequality, the inequalities

Definition 3

Definition 4

$$2x - 7 < x + 3$$

and

$$2x < x + 10$$

are equivalent.

Main Text

The equivalence preserving operations which we have deduced for inequalities, are (labelled Op (1) to Op (4) for reference):

Op (1) Add a constant to each side. **Op (1)**

Op (2) Subtract a constant from each side. **Op (2)**

Op (3) Multiply or divide each side by a positive constant. **Op (3)**

Op (4) Multiply or divide each side by a negative constant, and reverse the inequality. **Op (4)**

Exercise 6

Exercise 6
(3 minutes)

Which of the following statements are true?

The "allowable operations" Op (1) to Op (4) are defined on the set of

(i) all equations connecting real numbers;
(ii) all inequalities of the form $a < b$ where a and b are real numbers;
(iii) all inequalities of the form $a > b$ where a and b are positive real numbers;
(iv) all inequalities of the form $a < b$ or $a > b$ or $a \leqslant b$ or $a \geqslant b$ where a and b are any real numbers. ■

(continued on page 14)

Solution 4

Solution 4

Ineq (1) holds if $<$ is replaced by $\geqslant$, $\leqslant$ or $>$
Ineq (2) can be extended as follows:

If $a \geqslant b$ and $k \geqslant 0$, then $ka \geqslant kb$

If $a \leqslant b$ and $k \geqslant 0$, then $ka \leqslant kb$

If $a > b$ and $k > 0$, then $ka > kb$

Ineq (3) can be extended as follows:

If $a \geqslant b$ and $b > 0$, then $1/a \leqslant 1/b$

If $a \leqslant b$ and $b < 0$, then $1/a \geqslant 1/b$

If $a > b$ and $b > 0$, then $1/a < 1/b$

Ineq (4) can be extended as follows:

If $a \geqslant b$ and $k \leqslant 0$, then $ka \leqslant kb$

If $a \leqslant b$ and $k \leqslant 0$, then $ka \geqslant kb$

If $a > b$ and $k < 0$, then $ka < kb$ ■

Solution 5

Solution 5

(i) $a < 0, b < 0, a < b$:

In this case $a \cdot b$ is positive, by Exercise 6.1.2.1, and so by Ineq (2) we can multiply each side of the inequality $a < b$ by $\dfrac{1}{a \cdot b}$ (which must also be positive, as we saw in Deduction (3) on page 12) to give $\dfrac{1}{b} < \dfrac{1}{a}$, which is the same as $\dfrac{1}{a} > \dfrac{1}{b}$.

Thus if a and b are of the same sign and if $a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

(ii) $a < 0, b > 0$:

In this case, by necessity, $a < b$, and since $\dfrac{1}{a}$ is negative and $\dfrac{1}{b}$ is positive, $\dfrac{1}{a} < \dfrac{1}{b}$.

If $a = 0$ or $b = 0$, then the reciprocal has no meaning. ■

Solution 6

Solution 6

All these statements are true. (In the case of applying Op (4) to the set of all *equations* containing real numbers, the instruction to "reverse" may be ignored, owing to Eq (2).) ■

(*continued from page 13*)

Exercise 7

Exercise 7
(4 minutes)

State whether or not each of the following pairs of inequalities are equivalent.

(i)	$3x + 2 < 1$,	$3x < -1$
(ii)	$3x < -1$,	$x < -\frac{1}{3}$
(iii)	$-x < 4$,	$x < -4$
(iv)	$x^2(x - 3) < x^2(x + 4)$,	$x - 3 < x + 4$

■

Discussion

To solve an inequality (e.g. to reduce it to a form for which we can illustrate the solution set on the number line), we adopt much the same procedure as when solving an equation (but the steps must be carried out with considerably more care, for the allowable operations are more restricted).

For example, we can carry the solution of the inequality

$$2x - 7 < x + 3$$

which we considered earlier (page 13), a little further. We arrived at

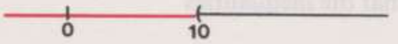
$$2x < x + 10.$$

If there *is* a real number x satisfying this inequality we can add $-x$ to each side and we get

$$x < 10$$

which is in a form which is simple enough to illustrate the solution set on the number line

$$\{x : x < 10\} = \{x : 2x - 7 < x + 3\}$$



The steps we followed in solving the inequality are closely analogous to the steps used in solving the corresponding equation:

$2x - 7 < x + 3$		$2x - 7 = x + 3$
$2x < x + 10$	(add 7)	$2x = x + 10$
$2x + (-x) < x + 10 + (-x)$	(add $-x$)	$2x + (-x) = x + 10 + (-x)$
$x < 10$		$x = 10$

Discussion ★★

If you are in doubt about how to solve an inequality, try solving the corresponding equation first, and then *try* to apply similar steps to the inequality.

In solving an equation, the object is to end up with a list of numbers each of which satisfies the equation. In solving an inequality, one aims to produce a list of intervals each point of which satisfies the inequality. For example, in the cases which we have just considered the solution of the equation is the number 10; the solution of the inequality is the interval consisting of all numbers less than 10. Just as an equation can have more than one number in its solution set, an inequality can have more than one interval in its solution set. These intervals can each be of one of the following types

$$\{x : x < a\}, \{x : x \leqslant a\}, \quad \{x : x > a\}, \{x : x \geqslant a\}$$

$$\{x : x < a \text{ and } x > b\}, \quad \{x : x \leqslant a \text{ and } x \geqslant b\}$$

$$\{x : x < a \text{ and } x \geqslant b\}, \quad \{x : x \leqslant a \text{ and } x > b\}$$

where a and b are any real numbers.

Notation ★★

The last four sets are usually written

$$\{x : b < x < a\}, \quad \{x : b \leqslant x \leqslant a\}$$

$$\{x : b \leqslant x < a\}, \quad \{x : b < x \leqslant a\}$$

or, more briefly

$$]b, a[, \quad [b, a]$$

$$[b, a[, \quad]b, a].$$

Notice the reversal of the bracket when the end-point of an interval is excluded from the solution set.

As we have seen, we can illustrate these intervals on the number line.

Exercise 8

Exercise 8
(4 minutes)

Illustrate on a number line the solution sets of the following inequalities.

(i) $4x + 1 < 3$,
(ii) $x^3 + 4x^2 > x^2$ (see part (iv) of Exercise 7),
(iii) $4 - 6x < 1$.

■

Solution 7 **Solution 7**

(i) YES, by Op (2).
(ii) YES, by Op (3), with $k = \frac{1}{3}$.
(iii) NO. Applying Op (4) with $k = -1$, we get $-x < 4$ and $x > -4$ as two equivalent inequalities.
(iv) NO. You may be tempted to apply Op (3) with $k = \dfrac{1}{x^2}$. But we cannot do this when $x = 0$, and indeed, while any real number x satisfies $x - 3 < x + 4$, the number 0 does not satisfy $x^2(x - 3) < x^2(x + 4)$. Multiplication by k is equivalence preserving *only if* k is a positive number. With $k = \dfrac{1}{x^2}$, k is not a positive number when $x = 0$, and this case must be dealt with separately. It follows that the inequalities

$$x - 3 < x + 4$$

and

$$x^2(x - 3) < x^2(x + 4)$$

are equivalent for $x \neq 0$, but, as we have seen, $x = 0$ does not satisfiy both inequalities and so must remain excluded. However, the inequalities

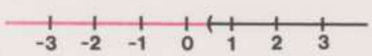
$$x - 3 \leqslant x + 4$$
$$x^2(x - 3) \leqslant x^2(x + 4)$$

are equivalent for $x \neq 0$. Looking at $x = 0$ separately, we see that $x = 0$ does satisfy both inequalities, and so they *are* equivalent for all values of x.

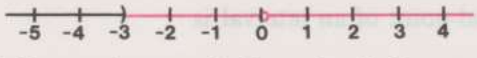
We remind you that two equations or inequalities may be equivalent even if one is not obtainable from the other by equivalence preserving operations — see page 10. ■

Solution 8 **Solution 8**

(i) $4x + 1 < 3$
is equivalent to $4x < 2$,
which is equivalent to $x < \frac{1}{2}$.

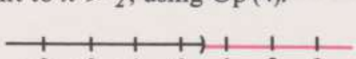


(ii) $x^3 + 4x^2 > x^2$
is equivalent to
$x^3 > -3x^2$,
which is equivalent to
$x > -3$ *provided x is not zero* (because $x^2 = 0$ if $x = 0$, $\dfrac{1}{x^2} > 0$ otherwise).
We now go back to test $x = 0$ independently, and we find that it does *not* satisfy the given inequality.



We put a circle round zero to indicate that it does not belong to the solution set.

(iii) $4 - 6x < 1$
is equivalent to $-6x < -3$,
which is equivalent to $x > \frac{1}{2}$, using Op (4).



■

6.2.2 The Combination of Inequalities

6.2.2
Main Text
* * *

We have been talking so far about carrying out operations on each side of an inequality. But can we combine inequalities in any way? For example, if $a < b$ and $c < d$, can we "add" the two inequalities (as we could if they were equations) and say that $a + c < b + d$, or "multiply" them and say that $ac < bd$?

We can prove that, if $a < b$ and $c < d$, then $a + c < b + d$, as follows:

> $b - a$ is positive and $d - c$ is positive, and therefore $(b + d) - (a + c)$, which equals $(b - a) + (d - c)$, is positive, and so $a + c < b + d$.

Thus, it is permissible to "add" two inequalities, provided they are both "less than" inequalities.

Exercise 1

Exercise 1
(3 minutes)

How is $a + c$ related to $b + d$ if

(i) $a < b$ and $c \leqslant d$,
(ii) $a \leqslant b$ and $c \leqslant d$,
(iii) $a > b$ and $c > d$,
(iv) $a > b$ and $c \geqslant d$,
(v) $a \geqslant b$ and $c \geqslant d$? ■

Exercise 2

Exercise 2
(2 minutes)

Can one validly "subtract" the inequalities

$a < b$ and $c < d$? ■

Exercise 3

Exercise 3
(2 minutes)

Investigate the statement

"if $a < b$ and $c < d$, then $ac < bd$"

(HINT: Try a few numerical examples.) ■

Solution 1

Solution 1

The solution of each part of this Exercise starts conveniently from the equation:

$$(b + d) - (a + c) = (b - a) + (d - c)$$

(i) Here $b - a$ is positive, and $d - c$ is positive or zero. In either case, their sum is positive. So the answer is:

$$a + c < b + d.$$

(ii) Here, $b - a$ is positive or zero, and $d - c$ is positive or zero. Their sum is zero if they are both zero, and is positive otherwise. So the answer is:

$$a + c \leqslant b + d.$$

(iii) Here, $b - a$ and $d - c$ are both negative, so their sum is negative. Thus:

$$a + c > b + d.$$

(iv) Here $b - a$ is negative, and $d - c$ is negative or zero. In either case, their sum is negative. Thus:

$$a + c > b + d.$$

(v) Here, $b - a$ is negative or zero, and $d - c$ is negative or zero. Their sum is zero if they are both zero, and negative otherwise. Thus:

$$a + c \geqslant b + d.$$

■

Solution 2

Solution 2

NO. For example, $2 < 3$ and $1 < 3$, but $(2 - 1) \not< (3 - 3)$. ■

Solution 3

Solution 3

The statement is true if all the numbers are positive — that is to say, if a and c are positive, for this automatically makes b and d positive. To prove this, we express $bd - ac$ in the following form:

$$\begin{aligned} bd - ac &= bd - ad + ad - ac \\ &= (b - a)d + a(d - c). \end{aligned}$$

Each term on the right-hand side is a product of two positive numbers, and is therefore positive. The whole expression is thus positive, and so

$$ac < bd.$$

If a, b, c and d are negative — this is to say, if b and d are negative — then the expression

$$a(d - c) + (b - a)d$$

is the sum of two terms each of which (by Exercise 6.1.2.1) is negative. Thus: $ac > bd.$

■

6.2.3 Simultaneous Inequalities

6.2.3

We frequently meet problems involving several inequalities. A system (set) of inequalities which are required to possess a common solution are called simultaneous inequalities. For example, if the solution set of one inequality is A and the solution set of another inequality is B, then the solution set of the two simultaneous inequalities is the set of elements which belong *both* to A and to B.

Definition 1
* * *

There is a less obvious occurrence of simultaneous inequalities which enables us to extend our technical facility for solving inequalities. It arises when we are confronted with having to solve an inequality such as

Discussion
* *

$$x^2 + x - 2 < 0$$

Inequality (1)

As Polya (*Polya** xvi, 98) tells us, one question which may usefully be asked, when confronted with a relatively new type of problem, is "Do we know a problem like it?" Well, in this case the answer is "Yes". We know the problem of finding the solution set of the equation

$$x^2 + x - 2 = 0$$

Equation (1)

which we tackle as follows. Using the properties of the real numbers which we have mentioned earlier, we reduce the equation to the equivalent equation

$$(x - 1)(x + 2) = 0\dagger$$

Equation (2)

At the next stage in the solution of Equation (2), we use an important, well-known property of the real numbers, which can be deduced from those properties which we already have, viz:

Main Text
* * *

Re (VI) *If a and b are real numbers such that $a \cdot b = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.*‡

Re (VI)

Applying this property to the solution of Equation (2) we have three possibilities:

$$(x - 1) = 0$$

Equation (3)

or

$$(x + 2) = 0$$

Equation (4)

or both

$$x - 1 = 0 \text{ and } x + 2 = 0 \text{ (which is obviously not possible).}$$

Finding the solution set of Equation (2) is therefore reduced to finding the solution sets of the two "alternative" equations, (3) and (4), each of which has a much simpler form. (Notice that neither (3) nor (4) is equivalent to (2); the solution set of (2) is equal to the set of elements which belong either to the solution set of (3) or to the solution set of (4). Also notice that (3) and (4) are not simultaneous equations. We shall be in a position to express these facts more concisely later in this text when we have developed a little of the notation of set algebra.) In general, we try to solve a quadratic equation like

See Glossary

$$ax^2 + bx + c = 0$$

* G. Polya, *How to Solve It*, Open University ed. (Doubleday Anchor Books, 1970). This book is the set book for the Mathematics Foundation Course; it is referred to in the text as *Polya*.

† You may be interested in the step by step derivation of Equation (2) from Equation (1), using only one property of R at each step, and stating each time the property you are using. As a start, we can for instance, write Equation (1) in the equivalent form

$$x^2 + 2x - x - 2 = 0$$

The steps are set out on page 21 if you are interested.

‡ You may like to try to deduce this property from the properties of R listed on page 1. The argument is set out on page 21, if you are interested.

by reducing it to two linear equations like

$$a_1x + b_1 = 0$$

$$a_2x + b_2 = 0$$

which we know how to solve.

But we also know how to solve linear inequalities like

$$ax + b < 0$$

Can we reduce an inequality like

$$cx^2 + dx + e < 0$$

to one or more linear inequalities?

The question we must ask ourselves is: What properties of R have we used to reduce Equation (1) to the Equations (3) and (4), and may we use these properties when dealing with inequalities?

Up to Equation (2) we are simply using the properties of addition and multiplication in R to rearrange the left hand side of the equation, and each of the steps will be equally valid for the Inequality (1). But the step from (2) to (3) and (4) relies on property Re (VI) which follows from the real number properties, without the ordering properties. But the properties of inequalities naturally invoke the properties of order, and so we *cannot* assume the analogue of Re (VI) for inequalities, namely that $ab < 0$ implies $a < 0$ or $b < 0$, or both.

So we can reduce the inequality

$$x^2 + x - 2 < 0$$

to the equivalent inequality

$$(x + 2)(x - 1) < 0,$$

but to see what the next stage is we must return to first principles. The type of inequality we are thinking about is

$$ab < 0$$

i.e. ab is negative.

Whether ab is negative or positive depends on whether a and b are positive or negative in the following way (see Exercise 6.1.2.1):

a	b	ab
+	+	+
+	−	−
−	+	−
−	−	+

(We have used + to stand for positive and − to stand for negative.) Thus ab is negative if one of a and b is positive and the other is negative, i.e. if

either $a < 0$ and $b > 0$
or $a > 0$ and $b < 0$

Inequalities (2)

(or both, which is not possible), which in our example take the form

either $x - 1 < 0$ and $x + 2 > 0$
or $x - 1 > 0$ and $x + 2 < 0$

(or both, which is again not possible). *(continued on page 21)*

OPTIONAL MATERIAL **Supplementary Material**

The steps in the argument showing that Equation (1) is equivalent to Equation (2).

$x^2 + x - 2 = 0$ **Equation (1)**

$\therefore^* \ x^2 + 1 \cdot x - 2 = 0$ (Property of 1 in Re (3))

$\therefore \ x^2 + (2 - 1) \cdot x - 2 = 0$

$\therefore \ x^2 + 2x - x - 2 = 0$ (Distributive property in Re (2))

$\therefore \ (x^2 - x) + (2x - 2) = 0$ (Commutative and associative properties in Re (2))

$\therefore \ (x \cdot x - 1 \cdot x) + (2 \cdot x - 2 \cdot 1) = 0$ (Property of 1 in Re (3))

$\therefore \ x \cdot (x - 1) + 2 \cdot (x - 1) = 0$ (Distributive property in Re (2))

$\therefore \ (x + 2) \cdot (x - 1) = 0$ (Distributive property in Re (2)) **Equation (2)**

Re (VI) *If a and b are real numbers such that $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both a and $b = 0$.* **Re (VI)**

Deduction (VI)

If $a \neq 0$, then Re (4), the equation

$a \cdot b = 0$

has a unique solution for b.

But $a \cdot 0 = 0$ by Re (II), so 0 is this solution.
Thus, if $a \neq 0$, then $b = 0$.
Similarly, if $b \neq 0$, then $a = 0$.
Thus, $a = 0$ or $b = 0$ or $a = b = 0$. ■

(*continued from page 20*)

The Inequality (1) is equivalent to the Inequalities (2). But what does (2) mean? It is quite a complicated statement.

Notice that in (2) the inequalities in the first line are "alternative" to the inequalities in the second (cf. "alternative" equations), and that the two inequalities on any one line are "simultaneous". A number will belong to the solution set of (2) if it belongs *either* to the set

$\{x : x - 1 < 0 \text{ and } x + 2 > 0\}$ i.e. $\{x : x < 1 \text{ and } x > -2\}$

or to the set

$\{x : x - 1 > 0 \text{ and } x + 2 < 0\}$ i.e. $\{x : x > 1 \text{ and } x < -2\}$

The first of these sets is the interval $]-2, 1[$. The second set contains no elements because no real number can at the same time be both >1 and <-2. Thus the solution set is the interval $]-2, 1[$.

Another way of tackling the type of inequality we have considered in the text above is to use a graphical approach.

* The symbol $\therefore$ means "therefore".

Exercise 1

Exercise 1
(5 minutes)

Using the same coordinates for each graph, sketch the graphs of the functions

$$f: x \longmapsto x + 2, \qquad (x \in R)$$

$$g: x \longmapsto x - 1. \qquad (x \in R)$$

(i) On your graph indicate the subset of the domain for which $g(x) > 0$ and $f(x) < 0$.
(ii) On your graph indicate the subset of the domain for which $g(x) < 0$ and $f(x) > 0$.
(iii) On your graph indicate that subset of the domain which is the solution set of the inequality $(x - 1)(x + 2) < 0$. ■

Exercise 2

Exercise 2
(3 minutes)

Find the solution set of the inequality

$$x^2 + x - 2 > 0. \quad ■$$

Some More Set Notation

Main Text
* * *

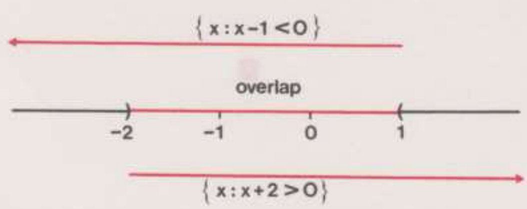
There is a notation for expressing inequalities like (2) in a more general way which is also very convenient when one is used to it. We shall be using this notation extensively in *Unit 11, Logic I* and also from time to time in the rest of the course.

We shall define two binary operations on sets, called intersection and union, which arise from the two types of statement making up Inequalities (2) on page 20.

The Intersection of Two sets

To be an acceptable candidate for membership of the set $\{x : x - 1 < 0$ and $x : x + 2 > 0\}$ an element must satisfy *both* conditions, or in other words it must belong to both of the sets

$$\{x : x - 1 < 0\} \quad \text{and} \quad \{x : x + 2 > 0\}$$



This "overlap" of two sets we call the intersection set.

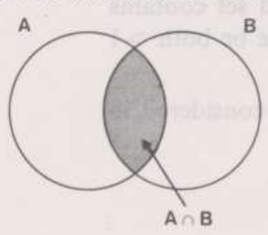
Definition 2
* * *

In general we define the intersection of two sets A and B as the set of all elements which belong both to A *and* to B. We denote this set by $A \cap B$, which we read as "A intersection B".

This definition applies to any two sets A and B, but our number line illustration applies only to sets of real numbers.

As another illustration, consider two sets of points in a plane, each of which is enclosed by a curve, and denoted in the diagram by A and B respectively.

The intersection of these two sets is the shaded region:



It is often convenient to use this type of diagram to illustrate more general sets. For example the region A could represent the set of all men and B the set of all Open University students. The shaded area would then represent the set $A \cap B$ which, in this case, would be the set of all male Open University students.

The Union of Two Sets

To be an acceptable candidate for membership of the solution set of Inequalities (2) on page 20, an element must belong to *one* (or possibly both) of the sets

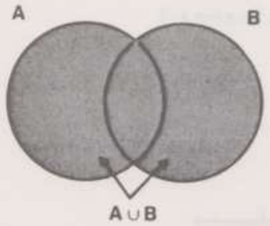
$$\{x : x - 1 > 0 \text{ and } x + 2 < 0\}$$

$$\{x : x - 1 < 0 \text{ and } x + 2 > 0\}$$

In general we define the union of two sets A and B to be the set of elements which belong to A *or* to B *or* to both A and B. We write this set as "$A \cup B$" which we read as "A union B".

Definition 3
* * *

The union of the two sets A and B shown in the previous diagram is shown shaded below.

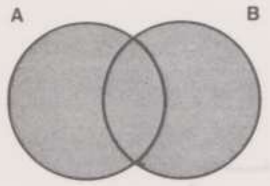


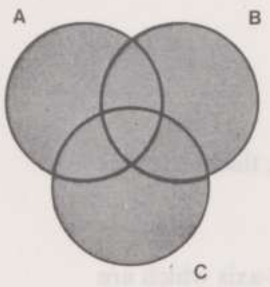
Union and Intersection as Operations

Main Text
* * *

If A and B are subsets of some set X, then so also are $A \cap B$ and $A \cup B$; in other words, $\cap$ and $\cup$ can be regarded as *binary operations* on the set of all subsets of X, analogous to the binary operations $\cdot$ and $+$ on R. How far does this analogy extend? We shall show below that it extends quite far, by setting out some properties of associativity, commutativity and distributivity. (We label these properties Set (1) to Set (4) for reference.)

Set (1) $\cup$ is both commutative and associative. **Set (1)**





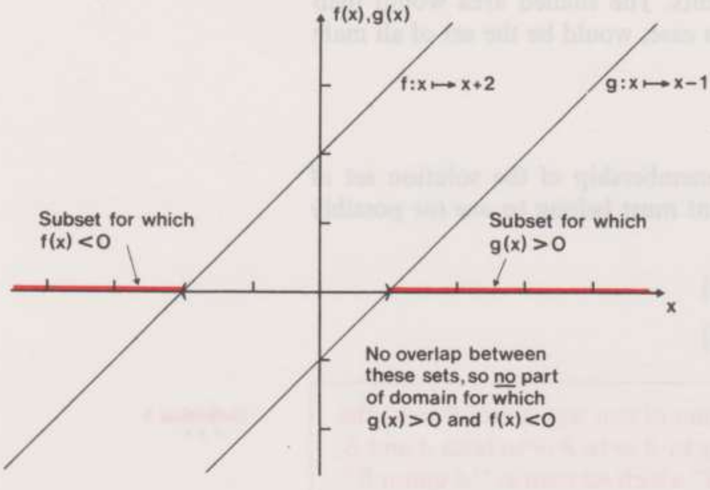
(The first diagram illustrates both $A \cup B$ and $B \cup A$. The second diagram illustrates both $A \cup (B \cup C)$ and $(A \cup B) \cup C$. The union of a collection

(*continued on page 26*)

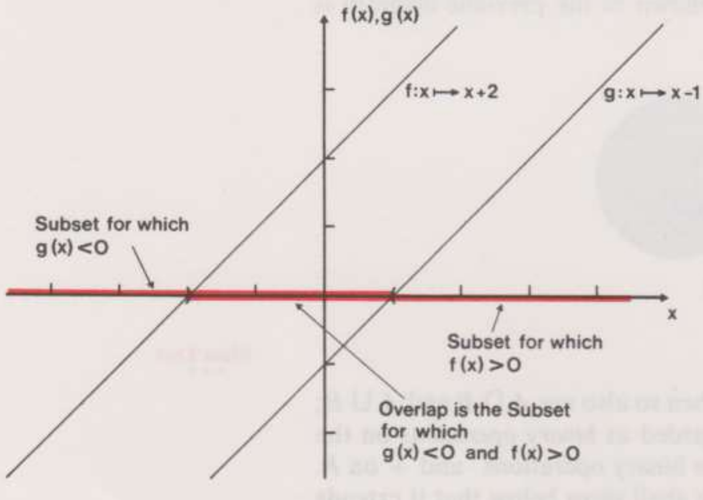
Solution 1

Solution 1

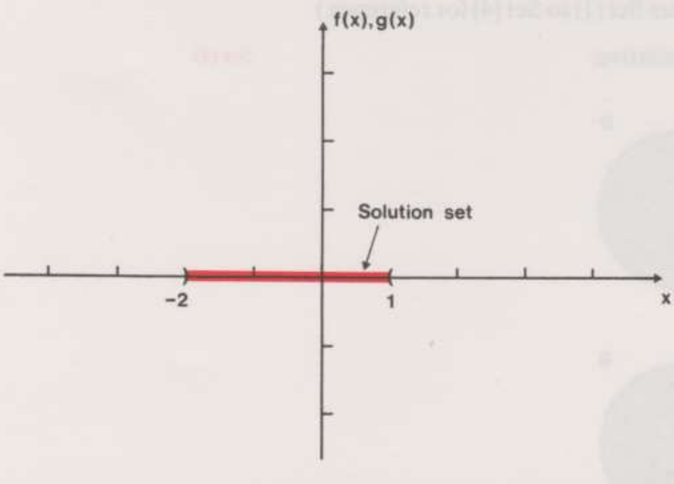
(i)



(ii)



(iii)



In (i), the required subset is the "overlap" of the red lines; there is in this case *no* overlap.

In (ii), the red lines do overlap.

In (iii), the required subset is the set of all points on the x-axis which are *either* in the subset determined in (i), *or* in that determined by (ii). This subset is shown in red; as there are no points at all in case (i), it is identical with the subset in case (ii). ■

Solution 2 **Solution 2**

If $(x - 1)(x + 2) > 0$, then

either $x - 1 > 0$ and $x + 2 > 0$

or $x - 1 < 0$ and $x + 2 < 0$

(Again, it is impossible for both to be true.) In graph (i) below, the subset of the x-axis for which

$x - 1 > 0$ and $x + 2 > 0$

is indicated. In graph (ii), the subset for which

$x - 1 < 0$ and $x + 2 < 0$

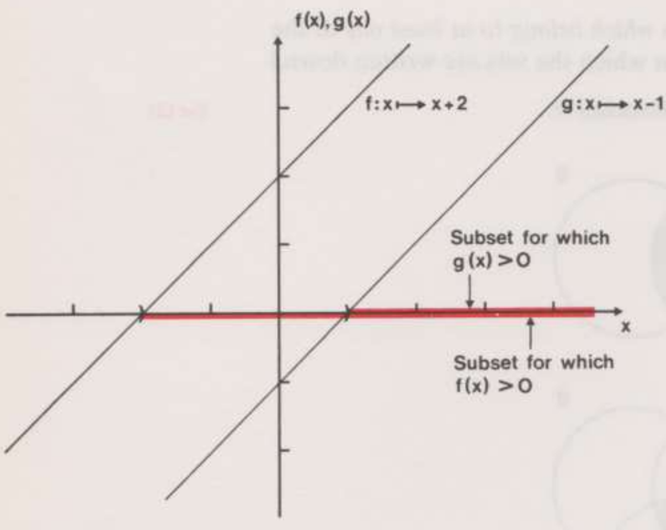
is indicated. In graph (iii), the solution set of

$x^2 + x - 2 > 0$

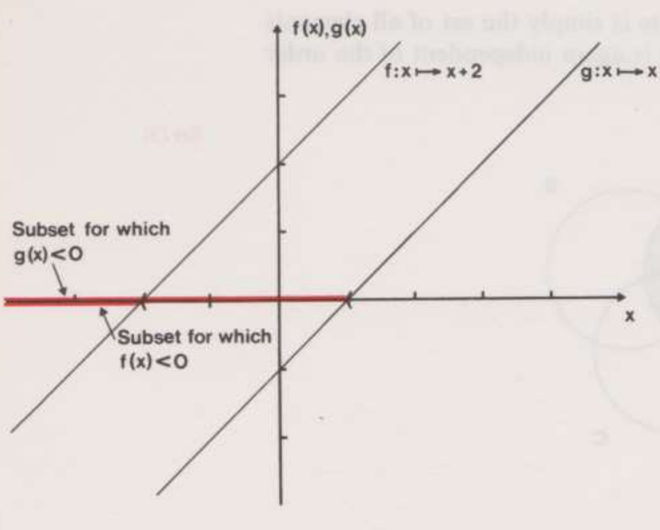
is indicated. It is, in fact, the set

$\{x \in R : x > 1 \text{ or } x < -2\}$ ■

(i)

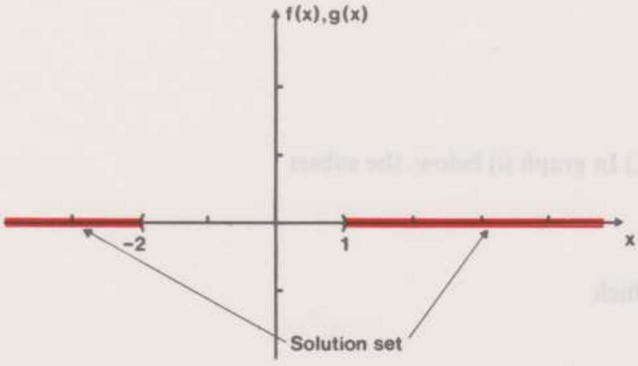


(ii)



Solution 2
(continued)

(iii)



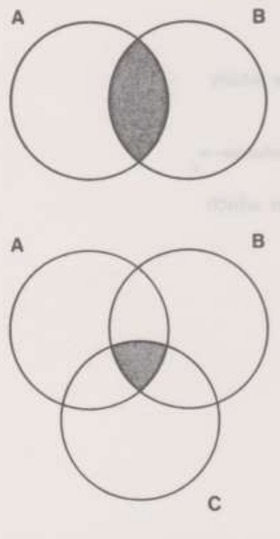
Note that in this case, in the statement of the solution "either A is true or B is true", there are *both* elements of the solution set for which A is true and B is not, *and* elements for which B is true and A is not. (A is the statement: $x + 2 > 0$ and $x - 1 > 0$; B is the statement: $x + 2 < 0$ and $x - 1 < 0$.) ■

(continued from page 23)

of sets is simply the set of all elements *which belong to at least one* of the sets, and is independent of the order in which the sets are written down.)

Set (2) $\cap$ is both commutative and associative.

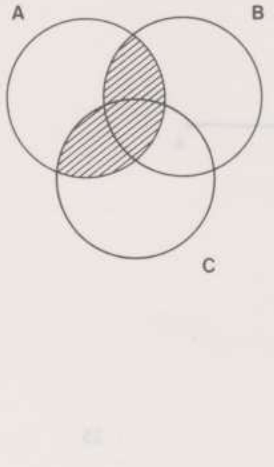
Set (2)

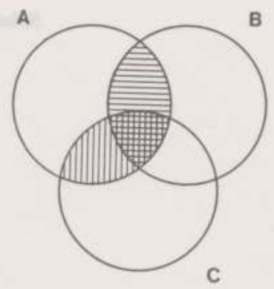


(The intersection of a collection of sets is simply the set of all elements *which are common to all* the sets, and is again independent of the order in which the sets are written down.)

Set (3) $\cap$ is distributive over $\cup$.

Set (3)





The first diagram represents $A \cap (B \cup C)$, and the second diagram shows $A \cap C$ in vertical shading and $A \cap B$ in horizontal shading. Clearly, the union of the shaded sets $A \cap C$, $A \cap B$ in the second diagram is equal to the shaded set in the first diagram.

In words, the argument can be put as follows:

$A \cap (B \cup C)$ is the set of all elements that are *both* in A *and* in $B \cup C$. That is, they are in A anyway, and also in B or C or both B and C. Thus, they are in A and in B, *or* they are in A and in C, *or* in A and in B and C. Therefore, they constitute the set $(A \cap B) \cup (A \cap C)$.

Exercise 3

Exercise 3
(3 minutes)

Use both diagrams and verbal argument to establish the following property.

Set (4) $\cup$ is distributive over $\cap$. ■

Set (4)

Exercise 4

Exercise 4
(2 minutes)

Describe in words the union of the set of all men and the set of all Open University students. ■

Exercise 5

Exercise 5
(3 minutes)

Illustrate the following sets

(i) $A = \{(x, y): x \in R, y \in R, x + y - 1 = 0\}$.
(ii) $B = \{(x, y): x \in R, y \in R, x^2 + y^2 - 1 = 0\}$.
(iii) $A \cap B$.
(iv) $A \cup B$. ■

Exercise 6

Exercise 6
(3 minutes)

(i) Illustrate the set $\{x: x - 1 > 0\} \cup \{x: x + 2 < 0\}$ on a number line.
(ii) Describe the set $\{x: x - 1 > 0\} \cap \{x: x + 2 < 0\}$. ■

Exercise 7

Exercise 7
(2 minutes)

Complete the following.

(i) For any set A, $A \cup \varnothing = \boxed{}$

(ii) For any set A, $A \cap \varnothing = \boxed{}$

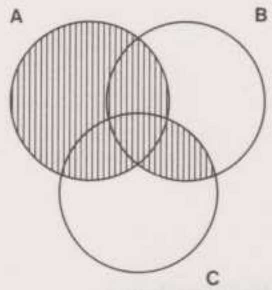
$\varnothing$ is the symbol for the empty set introduced on page 7. ■

We have spent a lot of time in this unit talking about solution sets, which are subsets of the set of real numbers. Whenever we define a subset, S, of a given set A, we automatically define another set, the set of all those elements of A which do *not* belong to S. In terms of the diagrams we have been using, and representing a proper subset of the set A by a region

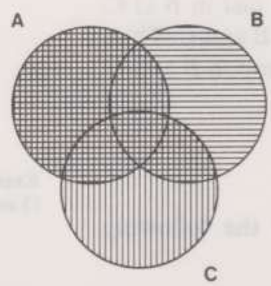
(continued on page 30)

Solution 3

Solution 3



$A \cup (B \cap C)$



Doubly shaded area represents
$(A \cup B) \cap (A \cup C)$

$A \cup (B \cap C)$ is the set of all objects that are *either* in A *or* in both B and C. Now any element in A is in $A \cup B$ and in $A \cup C$; and the same goes for any element that is both in B and in C; so certainly all elements of $A \cup (B \cap C)$ are in $(A \cup B) \cap (A \cup C)$. Conversely, any element of $(A \cup B) \cap (A \cup C)$ is *both* in $A \cup B$ *and* in $A \cup C$. Consequently, if it is not in A it must be in both B and C. So any element of $(A \cup B) \cap (A \cup C)$ is in $A \cup (B \cap C)$.

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. ■

Discussion
* *

The two distributive properties of the operations $\cap$ and $\cup$ for sets:

$$\left.\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}\right\} \text{ for all } A, B, C \in X$$

remind us of the distributive property of $\cdot$ over $+$ for real numbers:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \quad \text{for all } a, b, c \in R$$

But note that the operations $\cup$ and $\cap$ for sets are not completely analogous to the operations $\cdot$ and $+$ for real numbers, for

$$a + (b \cdot c) \neq (a + b) \cdot (a + c) \quad \text{for all } a, b, c \in R$$

i.e. $+$ is *not* distributive over $\cdot$, but $\cap$ *and* $\cup$ *are each distributive over the other.*

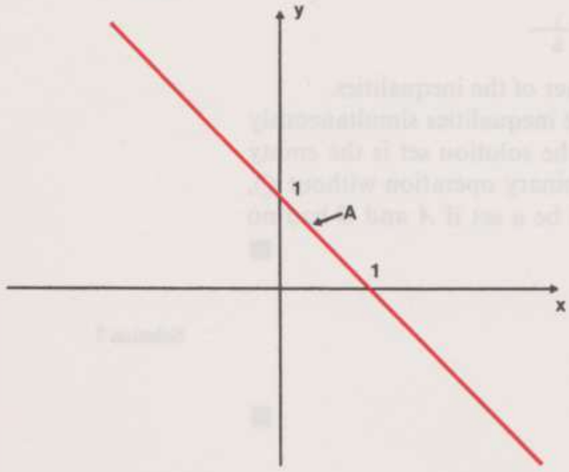
Solution 4

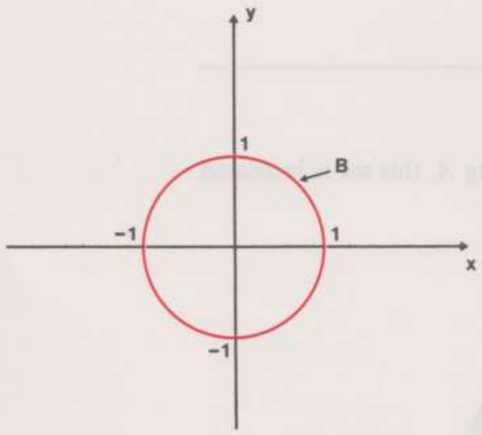
Solution 4

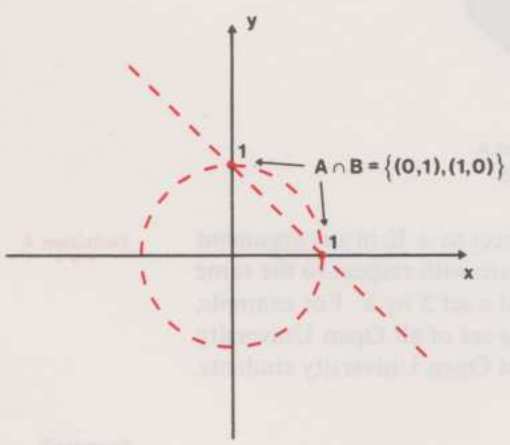
This is the set of all people who are *either* men *or* Open University students, including those men who are Open University students. ■

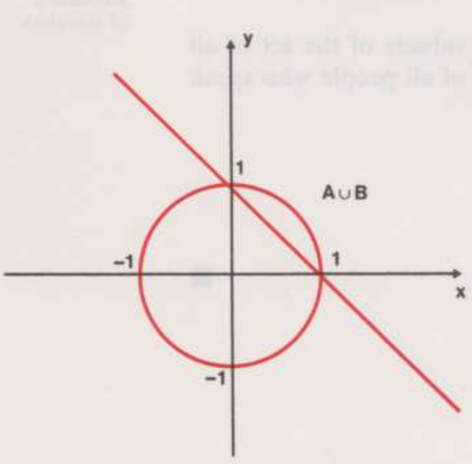
Solution 5

Solution 5





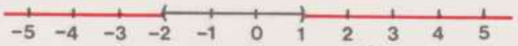




■

Solution 6

Solution 6

(i) 

Numbers between 1 and -2 satisfy neither of the inequalities.

(ii) There are no numbers satisfying both the inequalities simultaneously — the solution sets do not overlap. So the solution set is the empty set, $\varnothing$. (Notice that $\cap$ would not be a binary operation without $\varnothing$, because $A \cap B$ would not be defined to be a set if A and B had no elements in common.) ■

Solution 7

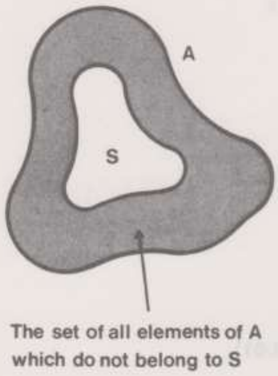
Solution 7

(i) $A \cup \varnothing = A$.

(ii) $A \cap \varnothing = \varnothing$. ■

(*continued from page 27*)

contained entirely within the region representing A, this set is indicated as follows:



This set is called the complement of S with respect to A. If, in any argument involving complements, all the complements are with respect to the same set, then we usually denote the complement of a set S by S'. For example, with respect to the set of all people, if S is the set of all Open University students, S' is the set of all people who are not Open University students.

Definition 4
***

Exercise 8

Exercise 8
(3 minutes)

All the following sets are to be considered as subsets of the set of all people. If A is the set of all males, B is the set of all people who speak English, describe in words the following sets:

(i) A'
(ii) B'
(iii) $(A')'$
(iv) $(A \cap B)'$
(v) $A' \cup B'$ ■

6.2.4 Some Non-linear Inequalities

6.2.4

The only type of inequality which we have been able to solve generally so far is

Definitions

$$ax + b < 0$$

with a and b given numbers and $a \neq 0$.

Such an inequality is called a linear inequality in one unknown. Any other type of inequality in one unknown is said to be non-linear.

Definition 1
Definition 2

In 6.2.3 we discussed the particular non-linear inequality

$$(x - 1)(x + 2) < 0.$$

We can write out the solution of this inequality in a formal way, in terms of solution sets. This shows how a verbal argument can be expressed symbolically. We shall be discussing the representation of logical arguments in algebraic form in *Unit 17, Logic II*.

The inequality $(x - 1)(x + 2) < 0$ is satisfied if

either

$$x - 1 < 0 \quad \text{and} \quad x + 2 > 0$$

or

$$x - 1 > 0 \quad \text{and} \quad x + 2 < 0,$$

so we may write

$$\begin{aligned} &\{x : (x - 1)(x + 2) < 0\} \\ &= \{x : x - 1 < 0 \textit{ and } x + 2 > 0\} \cup \{x : x - 1 > 0 \textit{ and } x + 2 < 0\} \\ &= (\{x : x - 1 < 0\} \cap \{x : x + 2 > 0\}) \cup (\{x : x - 1 > 0\} \cap \{x : x + 2 < 0\}) \\ &= (\{x : x - 1 < 0\} \cap \{x : x + 2 > 0\}) \cup \varnothing \qquad \text{(see p. 30)} \\ &= \{x : x - 1 < 0\} \cap \{x : x + 2 > 0\} \qquad \text{(see p. 30)} \\ &= \{x : x < 1\} \cap \{x : x > -2\} \\ &= \{x : -2 < x < 1\} \\ &=]-2, 1[\end{aligned}$$

Exercise 1

Exercise 1
(3 minutes)

Illustrate on a number line the solution sets of the following inequalities:

(i) $(x + 3)(x - 1) < 0$,
(ii) $(x + 2)(x + 3) > 0$. ■

The next exercise introduces a different method of solving inequalities.

Exercise 2

Exercise 2
(3 minutes)

Draw the graph whose equation is

$$y = (x - 4)(x + 1)$$

Indicate that part of the x-axis which corresponds to $y < 0$. Write down the solution set of the inequality

$$(x - 4)(x + 1) < 0. \quad ■$$

(continued on page 32)

Solution 6.2.3.8

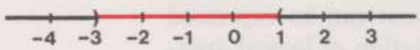
Solution 6.2.3.8

(i) The set of all females.
(ii) The set of all people who do not speak English.
(iii) The set of all males.
(iv) The set of all people who are not both male and English-speaking. (Alternatively: the set of all people who are either female or non-English-speaking or both.)
(v) The same as (iv). ■

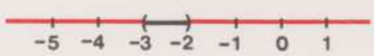
Solution 1

Solution 1

(i)



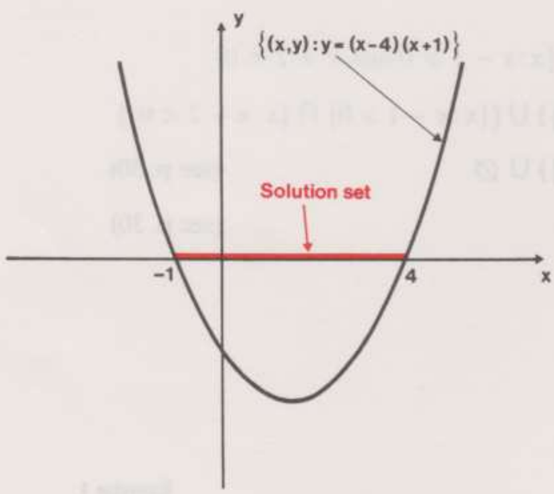
(ii)



■

Solution 2

Solution 2



The solution set is $\{x: -1 < x < 4\}$. ■

(*continued from page 31*)

Discussion

This last exercise demonstrates how inequalities can sometimes be solved graphically. If we can draw the graph whose equation is $y = f(x)$ then the solution of the inequality $f(x) < 0$ is equivalent to locating those regions where the curve drops below the x-axis. The examples we have considered so far have all been cases where the graph of f is easy to draw. In other cases a more ingenious method may be required. (For the examples we have met so far, both the graphical method and the method of logical argument require the solution of the equation $f(x) = 0$.)

6.2.5 Summary

6.2.5

Summary

In the middle section of this unit we have developed some techniques for manipulating inequalities.

The procedure for solving an inequality is much the same as that for solving an equation. By carrying out a sequence of operations on each side of an inequality which leave the solution set unchanged, we try to reduce the inequality to a form for which the solution set is easily recognized. These operations were discussed in 6.2.1 and enable us to solve any "linear" inequality.

We then discussed the situation which arises when we have two or more inequalities. Two questions arose. Can we combine the inequalities in any way? Can we find a set of numbers which satisfy both inequalities simultaneously? To answer the second question we found it convenient to extend our vocabulary by introducing a few more ideas about sets. This short piece is really a side track in the main development of the unit — but we shall return to the topic later in *Unit 11, Logic I.*

Simultaneous inequalities find immediate application, for we can extend our technical scope by regarding certain non-linear inequalities as sets of simultaneous or alternative inequalities.

If you feel you would like more practice, you may like to have a look at one of the books mentioned in the bibliography. But it would be as well to wait until you have worked through the unit to see whether you have enough time.

6.3 INEQUALITIES ARISING FROM MAPPINGS OF $R \times R$ TO R

6.3

6.3.1 Solution Sets as Subsets of the Plane

6.3.1

Discussion

So far we have been discussing inequalities which can be put in the form $f(x) > 0$, where f is a function with domain and codomain which are subsets of R, i.e. a function of one (real) variable. Finding the solution set of such an inequality is equivalent to starting with the set R (which we have represented by a number line) and interpreting the inequality as a condition which selects a subset of R (or a portion or collection of portions of the number line). The solution process is equivalent to an identification of that subset.

But we can also consider problems involving functions from $R \times R$ to R (i.e. functions of two (real) variables) because, although we have not discussed the possibility of ordering the domain (and so we cannot talk of inequalities between elements of the domain), it is not the domain which *has* to be ordered. Inequalities are expressed in terms of the codomain. For example, if functions f and g are defined by:

$$f:(x, y) \longmapsto x + y \qquad ((x, y) \in R \times R)$$

and

$$g:(x, y) \longmapsto 3 \qquad ((x, y) \in R \times R)$$

then the inequality

$$f(x, y) < g(x, y)$$

certainly is meaningful, and is equivalent to

$$x + y < 3 \qquad ((x, y) \in R \times R)$$

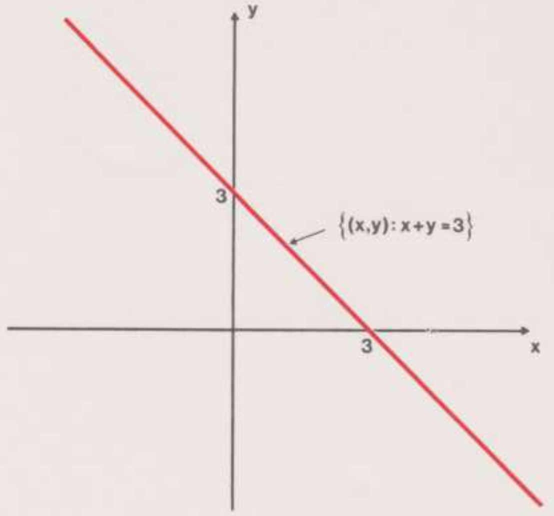
We know that, just as we can represent a single real number as a point on a number line, we can represent a pair of real numbers (x, y) as a point in a plane. If there is no restriction on x and y other than that they must be real, then the set of all pairs (x, y) under consideration is represented by a plane. We have seen all this before in *Unit 3, Operations and Morphisms.*

Any further restriction on x or y or both will restrict the point with coordinates (x, y) to a subset of the plane.

A simple case would be the requirement

$$x + y = 3$$

Then, instead of ranging over the entire plane, the point with coordinates (x, y) is restricted to be on a line.



The line can be regarded as representing the solution set of the equation $x + y = 3$.

Exercise 1

Exercise 1
(2 minutes)

The line with equation $x + y = 3$ splits the plane into three subsets. In one subset we have $x + y < 3$, in another $x + y > 3$, and in the third $x + y = 3$. By choosing a few points in the plane, verify this statement and identify the three sets. ■

Exercise 2

Exercise 2
(2 minutes)

Illustrate the solution set of $2x - 3y < 4$. ■

6.3.2 Some More Non-linear Inequalities

6.3.2

Definition 1

The inequalities discussed in the last section are called linear: the boundary of the solution set is a straight line. Any inequality of the form $ax + by + c < 0$ is of this type. But we are not restricted to lines only.

Exercise 1

Exercise 1
(2 minutes)

The graph with equation

$$x^2 + y^2 = 16$$

is a circle. Draw this circle and indicate the region where

$$x^2 + y^2 < 16.$$ ■

Exercise 2

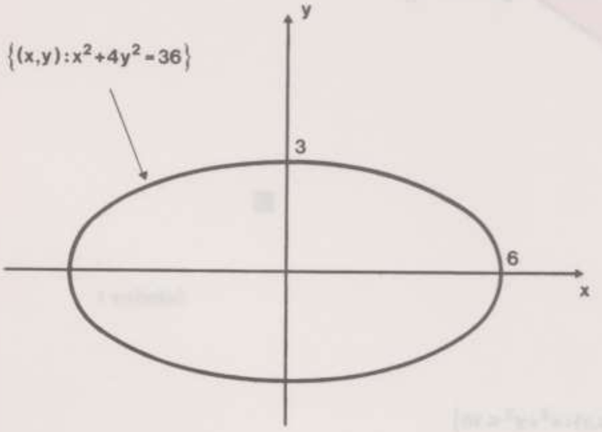
Exercise 2
(2 minutes)

The equation

$$x^2 + 4y^2 = 36$$

is that of the ellipse shown:

See Glossary



Illustrate the solution sets of the following inequalities:

(i) $x^2 + 4y^2 < 36$,
(ii) $x^2 + 4y^2 > 36$. ■

Exercise 3 (Be careful!)

Exercise 3
(2 minutes)

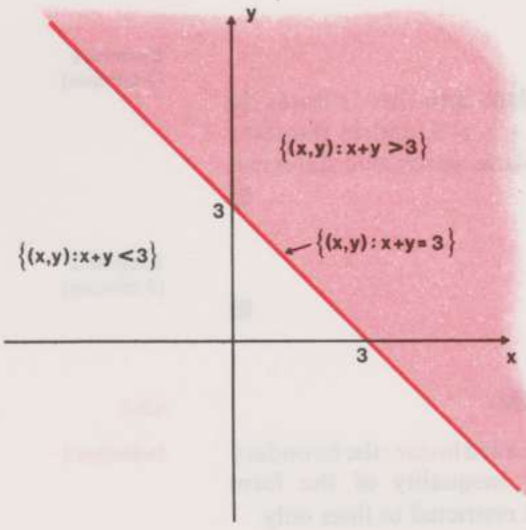
The graph with equation

$$(ax + by + c)^2 = 0$$

is a straight line. Draw this line and indicate the region of the plane where $(ax + by + c)^2 > 0$. ■

Solution 6.3.1.1

Solution 6.3.1.1



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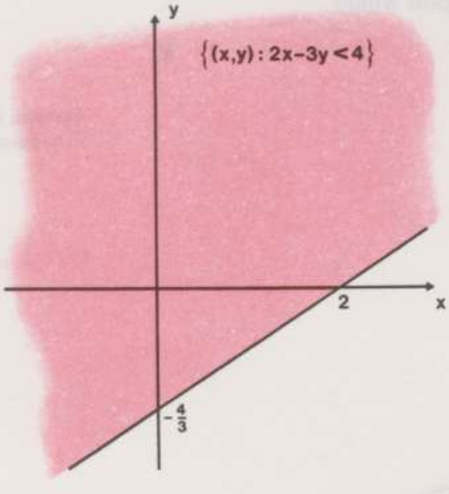
Solution 6.3.1.2

Solution 6.3.1.2



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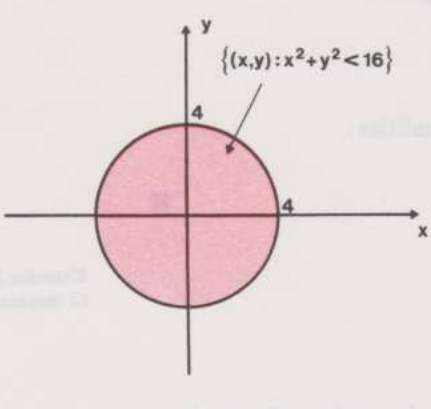
Solution 1

Solution 1



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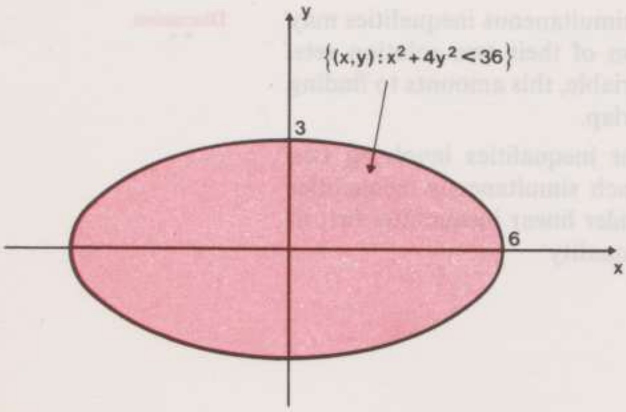
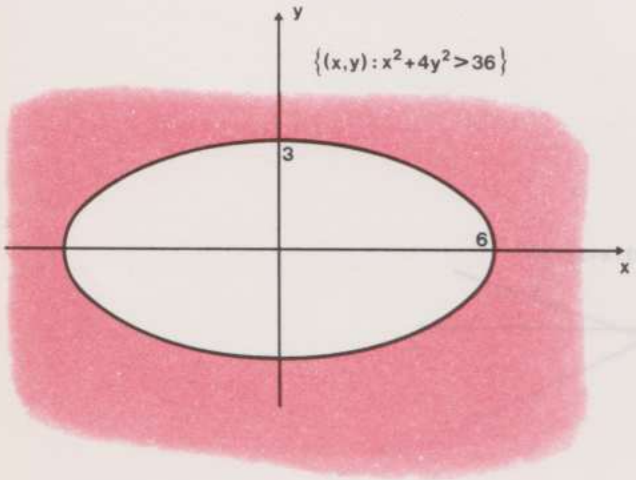
Solution 2

Solution 2





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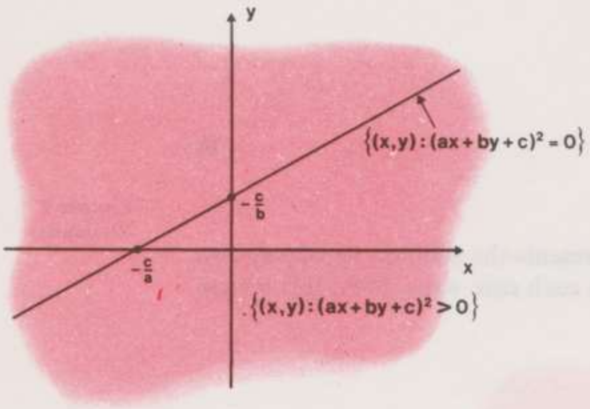
Solution 3

Solution 3



The region $(ax + by + c)^2 > 0$ consists of every point in the plane *not* lying on the line $ax + by + c = 0$. ■

6.3.3 Simultaneous Inequalities

6.3.3

Discussion

We have seen that the solution set of two simultaneous inequalities may be obtained by calculating the intersection of their two solution sets. In the case of inequalities involving one variable, this amounts to finding where two portions of the number line overlap.

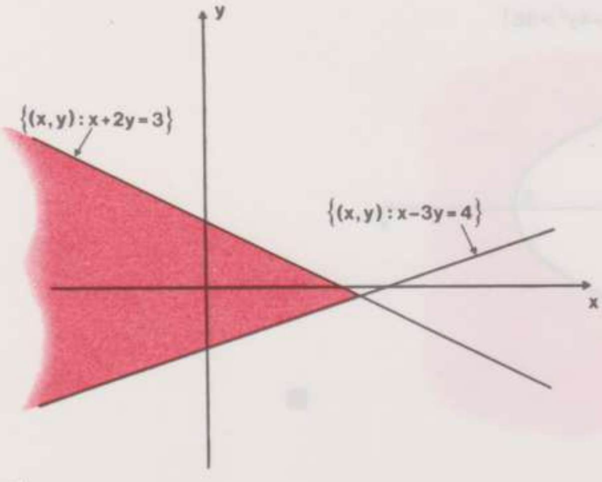
One would expect a similar situation for inequalities involving two variables, and we can indeed deal with such simultaneous inequalities without introducing any more ideas. Consider linear inequalities first of all. For example, the solution set of the inequality

$$x + 2y < 3$$

is a "half-plane", and the solution set of

$$x - 3y < 4$$

is another half-plane. Both inequalities will be satisfied in the region where these half planes overlap:



Exercise 1

Exercise 1
(3 minutes)

Illustrate the region where the inequalities

$$x + 2y < 3$$
$$x - 3y < 4$$
$$x + y < 0$$

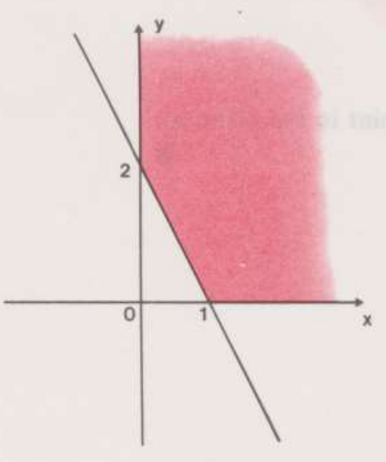
are satisfied simultaneously. ■

Exercise 2

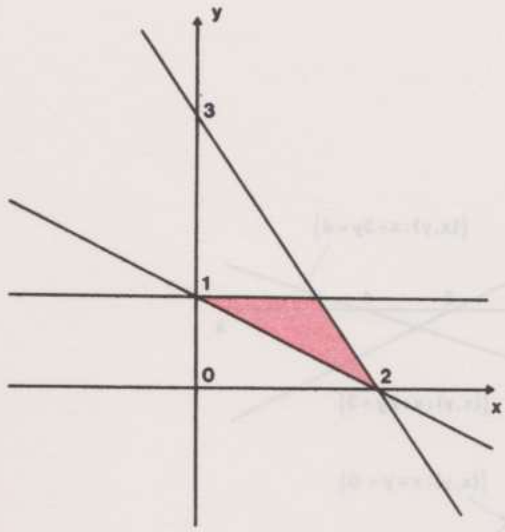
Exercise 2
(5 minutes)

Each of the shaded regions below represents the solution set of a system of simultaneous linear inequalities. In each case write down this system of inequalities.

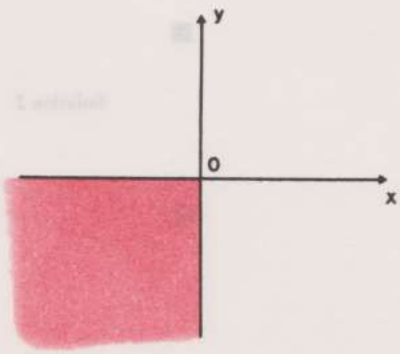
(i)



(ii)



(iii)



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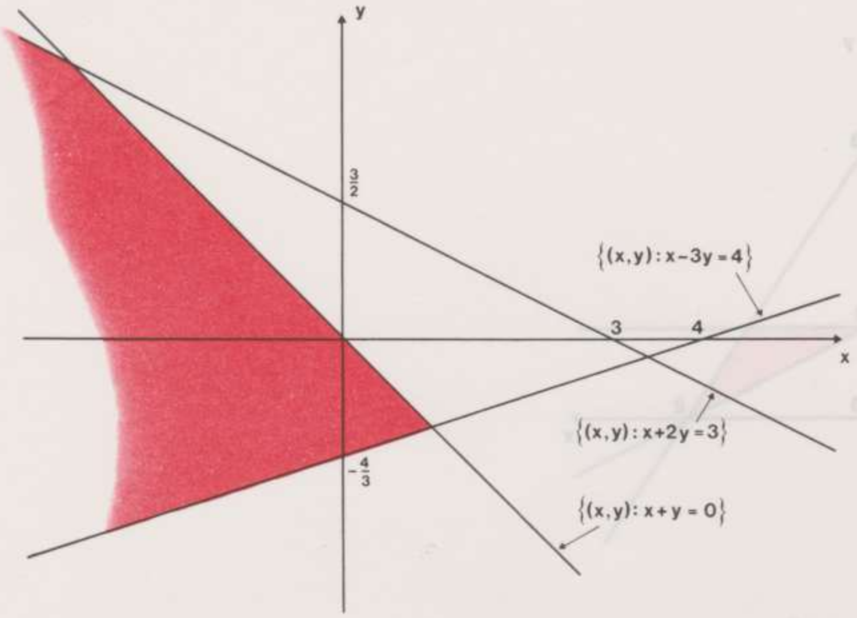
Solution 1

Solution 1



■

Solution 2

Solution 2

(i) $x > 0$, $y > 0$, $2x + y > 2$.
(ii) $y < 1$, $x + 2y > 2$, $3x + 2y < 6$.
(iii) $x < 0$, $y < 0$.

■

6.3.4 An Application

6.3.4

Application

In this section we shall indicate how some of the ideas we have been discussing can be applied to a type of problem which arises frequently in management.

A typical example is that of a mine owner who has various options open to him for exploiting his mine. He can exhaust the mine by producing a constant output; he can start with a small output and gradually increase production; he can start with a high production and gradually reduce production; there are many possibilities. Given some economic predictions of prices, costs and so on he will presumably wish to arrange his production to give the largest profit.

A manufacturer of ladies' coats may find that he can make a bigger profit on a winter coat than a summer coat, but summer coats are easier to sell. In any case, he will presumably be working under certain constraints, such as commitments to supply a certain number of each type of coat and maximum possible output, because of the number of machines and workers he has at his disposal. So again, he has a variety of possible production arrangements available to him and he can choose from these the one that suits him best; the one that maximizes profit, or minimizes cost, or diminishes hard work on his part, or whatever criterion he decides on.

Problems of this type fall into two parts. First, one has to determine the set of possible production arrangements, and then out of this set choose the optimum arrangement.

Exercise 1

Exercise 1
(10 minutes)

A farmer makes two types of fertilizer, X and Y, using chemicals A and B. Fertilizer X is composed of 75% of chemical A and 25% of chemical B. Fertilizer Y is composed of 50% of chemical A and 50% of chemical B. He requires at least 40 cwt of X and at least 60 cwt of Y, and has available 100 cwt of A and 60 cwt of B. If he makes x cwt of X and y cwt of Y, find four inequalities satisfied by x and y.

(HINT: One of the inequalities can be obtained by writing down how much of chemical A is used to make x cwt of X and y cwt of Y; another can be obtained by considering how much of chemical B is used.) ■

Exercise 2

Exercise 2
(2 minutes)

Illustrate the region representing the set of pairs (x, y) of all possible production arrangements in the previous exercise. ■

Exercise 3

Exercise 3
(3 minutes)

If the farmer wishes to make as much fertilizer as possible, he must find the pair (x, y) belonging to the set you illustrated in Exercise 2 for which $x + y$ is maximum. Find this pair. ■

Exercise 4

Exercise 4
(5 minutes)

A manufacturer of tinned food produces "Beans with Minced Beef" which consists of 90% beans and 10% mince, and "Minced Beef with Beans" which consists of 50% beans and 50% mince. His market researchers tell him that each week he should produce at least 150 cwt of "Beans with Mince" and at least 100 cwt of "Mince with Beans". However his suppliers can only supply him each week with 270 cwt of beans and 100 cwt of mince. If he produces x cwt of "Beans with Mince" and y cwt of "Mince with Beans" write down four inequalities satisfied by x and y and illustrate the possible pairs (x, y) from which he can choose. (Ignore possible sophistications such as carrying over a stock of beans to the following week.) ■

Exercise 5

Exercise 5
(3 minutes)

If the manufacturer in Exercise 4 makes 2p profit on a tin of "Beans with Mince" and 3p profit on a tin of "Mince with Beans", how should he arrange his production to make the maximum profit? ■

Solution 1 **Solution 1**

$$\tfrac{3}{4}x + \tfrac{1}{2}y \leqslant 100$$

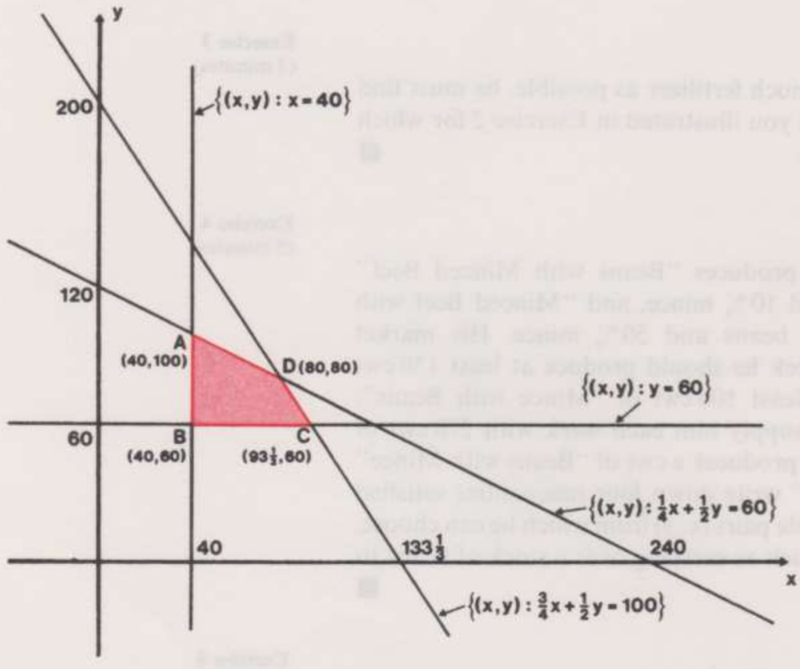
$$\tfrac{1}{4}x + \tfrac{1}{2}y \leqslant 60$$

$$x \geqslant 40$$

$$y \geqslant 60$$

■

Solution 2 **Solution 2**



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Solution 3 **Solution 3**

$$x = 80, \quad y = 80$$

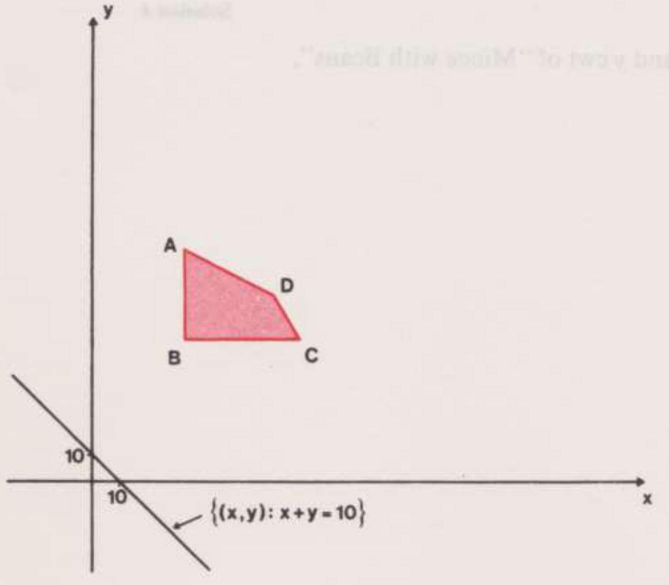
In the above diagram, the maximum value of $x + y$ occurs at D. If you got this answer for the maximum value, how did you do it? By guesswork? Because it was "obvious"? How would you do it in a more complicated case? Even more pressing, if you did not get the right answer you probably want to know how you can get it.

We want to find the maximum value of $x + y$, so let us write c for $x + y$, i.e.:

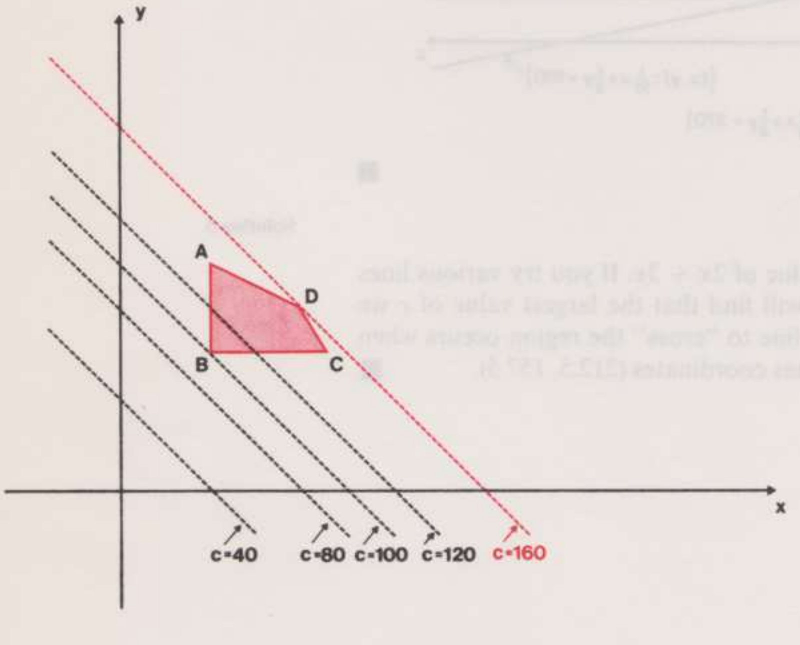
$$x + y = c$$

For a given value of c this is the equation of a straight line: for example, when $c = 10$ we get the line with equation $x + y = 10$.

Solution 3
(continued)



This line does not cross the region *ABCD*, so there is no pair (x, y) in this region which makes $x + y = 10$. We are trying to find the maximum value of c, so let us try a few more values of c. Notice that we shall get a set of parallel lines:



When $c = 100$, the line passes through B and so 100 is a possible value for $x + y$: at B, $x = 40$ and $y = 60$. When $c = 120$, the line passes through the region, and every point which is both on the line and in the region represents a possible arrangement giving 120 cwt of fertilizer. We can solve the problem by continuing to increase c until the line is just leaving the region. If we do this we find that the maximum value of c occurs when the line passes through the point D, representing a total production of 160 cwt of fertilizer. ■

Solution 4

Solution 4

To make x cwt of "Beans with Mince" and y cwt of "Mince with Beans", he needs

$$\tfrac{9}{10}x + \tfrac{1}{2}y \quad \text{cwt of beans}$$

and

$$\tfrac{1}{10}x + \tfrac{1}{2}y \quad \text{cwt of mince.}$$

Thus,

$$\tfrac{9}{10}x + \tfrac{1}{2}y \leqslant 270$$

and

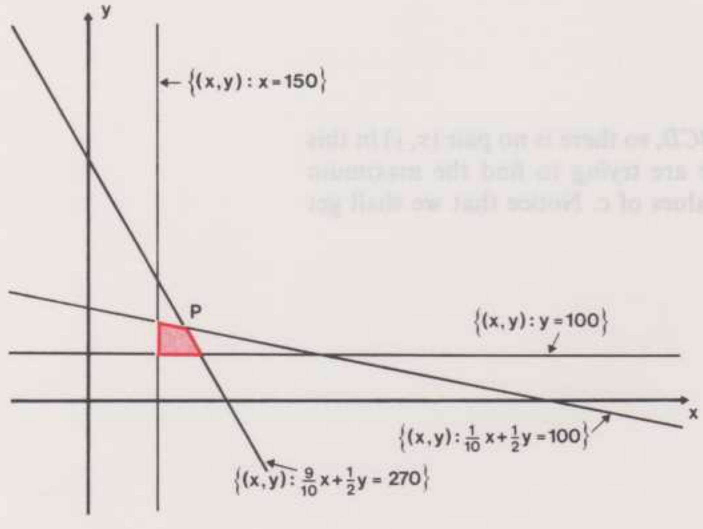
$$\tfrac{1}{10}x + \tfrac{1}{2}y \leqslant 100.$$

Also we know that

$$x \geqslant 150$$

and

$$y \geqslant 100$$



■

Solution 5

Solution 5

We have to find the maximum value of $2x + 3y$. If you try various lines with equation $2x + 3y = c$, you will find that the largest value of c we can have while still requiring the line to "cross" the region occurs when the line passes through P, and P has coordinates (212.5, 157.5). ■

Discussion

The examples which we have been discussing are obviously greatly oversimplified. There is little point in trying to make them realistic for the technical details would overshadow the principles involved. In real life there may be a hundred or more variables and hundreds of inequalities. But the principles are the same; one has to maximize or minimize an expression involving a set of variables which are constrained to satisfy a set of conditions. Sophisticated methods have been devised both for setting up the problem and for its solution, but that is not our concern here.

Definition 1

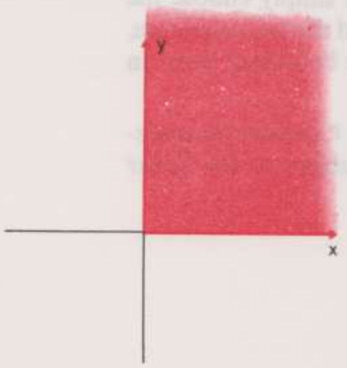
The problems are examples of Linear Programming problems but the name is not important. (As Sherlock Holmes said: "You mentioned your name as if I should recognize it, but beyond the obvious facts that you are a bachelor, a solicitor, a Freemason, and an asthmatic, I know nothing about you.")

In each of the examples we had to maximize a *linear* expression subject to *linear* inequalities and we found that the linear expression attains a maximum value at one of the vertices of the solution set of the inequalities.

This is one of three possibilities which may arise. The other two are:

(i) The maximum value may exist along a boundary. This will occur when the "profit line" is parallel to one of the edges of the region.

(ii) The "permissible set" may not be bounded. It may look like this!



In the first case our remarks still apply — the maximum value still occurs at a vertex, but not only at a vertex. In the second case, the problem is not well defined — there *is* no maximum value for certain linear expressions, such as $2x + y$, for instance (though in this case a linear expression such as $-2x - 3y$, for example, will have a maximum value at the point $(0, 0)$ in the "permissible set").

Exercise 6

Exercise 6
(3 minutes)

Suggest a plausible argument for our conclusion that a linear expression will have a maximum value at one of the vertices of the solution set of the inequalities, or along one of the boundary lines (provided that it has a maximum anywhere at all in the solution set). ■

Solution 6

Solution 6

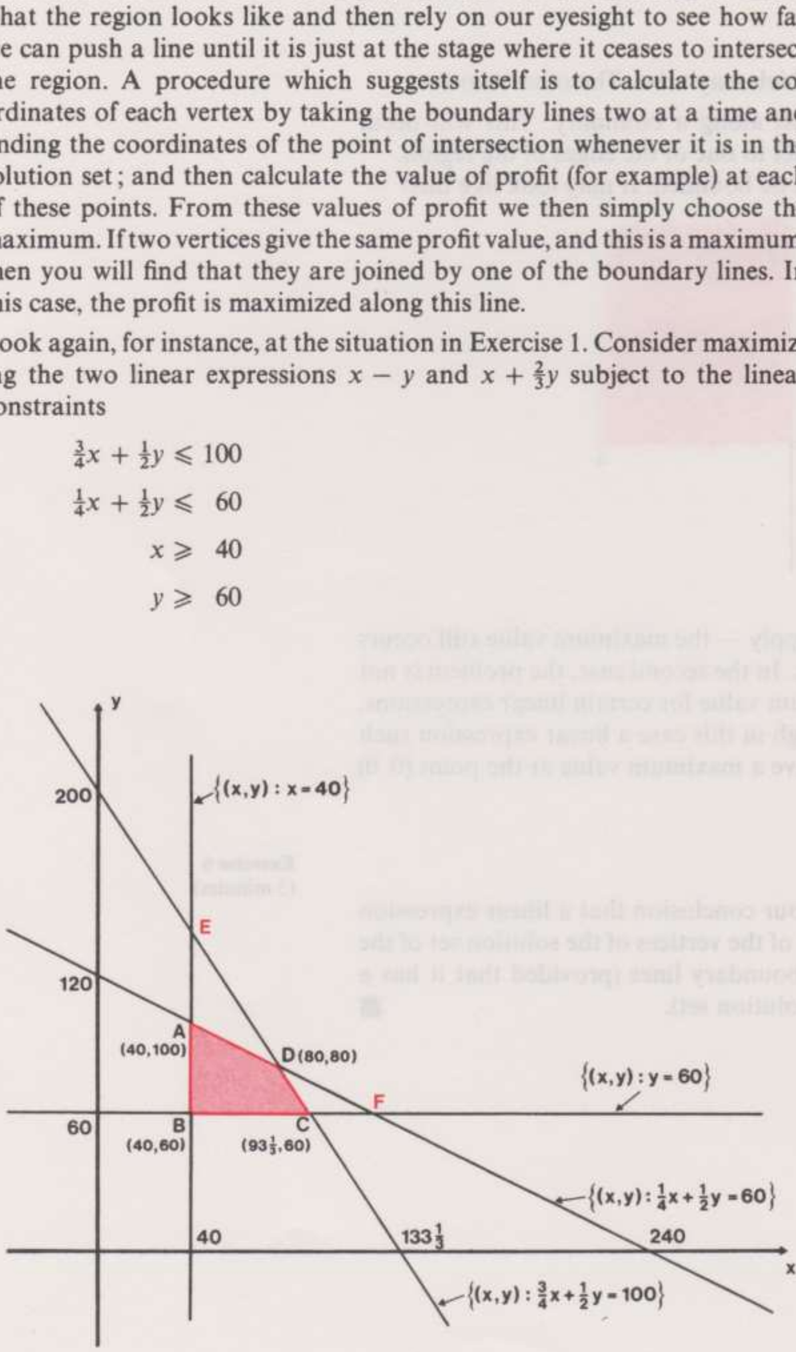
If a point in the solution set is not either on a vertex or on a boundary line of the solution set, then it is in the "interior" of the solution set, in the sense that it is "surrounded on all sides" by other points in the solution set. In this case, since the expression in question is linear, some of the points surrounding it and in the solution set will give a *greater* value for the expression (and some, a *lesser* value). So the point in question cannot give a maximum value to the expression, among points in the solution set. Hence, any point that *does* give a maximum *must* lie on the "boundary" of the solution set, i.e. on a vertex or a boundary line. ■

Discussion

So if the problem is properly defined, we will find the solution at a vertex or on a boundary line. Now each vertex of the region is the point of intersection of two of the boundary lines. So we do not need to know what the region looks like and then rely on our eyesight to see how far we can push a line until it is just at the stage where it ceases to intersect the region. A procedure which suggests itself is to calculate the co-ordinates of each vertex by taking the boundary lines two at a time and finding the coordinates of the point of intersection whenever it is in the solution set; and then calculate the value of profit (for example) at each of these points. From these values of profit we then simply choose the maximum. If two vertices give the same profit value, and this is a maximum, then you will find that they are joined by one of the boundary lines. In this case, the profit is maximized along this line.

Look again, for instance, at the situation in Exercise 1. Consider maximizing the two linear expressions $x - y$ and $x + \frac{2}{3}y$ subject to the linear constraints

$$\begin{aligned} \tfrac{3}{4}x + \tfrac{1}{2}y &\leqslant 100 \\ \tfrac{1}{4}x + \tfrac{1}{2}y &\leqslant 60 \\ x &\geqslant 40 \\ y &\geqslant 60 \end{aligned}$$



The values of $x - y$ and $x + \frac{2}{3}y$ at the points E and F do not need to be calculated, since E and F are not in the solution set. As for the others:

At A, $x - y = -60$ and $x + \frac{2}{3}y = 106\frac{2}{3}$

At B, $x - y = -20$ and $x + \frac{2}{3}y = 80$

At C, $x - y = 33\frac{1}{3}$ and $x + \frac{2}{3}y = 133\frac{1}{3}$

At D, $x - y = 0$ and $x + \frac{2}{3}y = 133\frac{1}{3}$

Thus, $x - y$ is maximized at C, and $x + \frac{2}{3}y$ is maximized along the line joining C and D.

Exercise 7

Exercise 7
(5 minutes)

Solution sets which are not bounded by straight lines are not so easy to deal with. Sketch the region representing $A \cap B \cap C$ where:

$$A = \{(x, y): y \leqslant -(x - 3)(x + 1)\}$$

$$B = \{(x, y): y \geqslant 1 - x\}$$

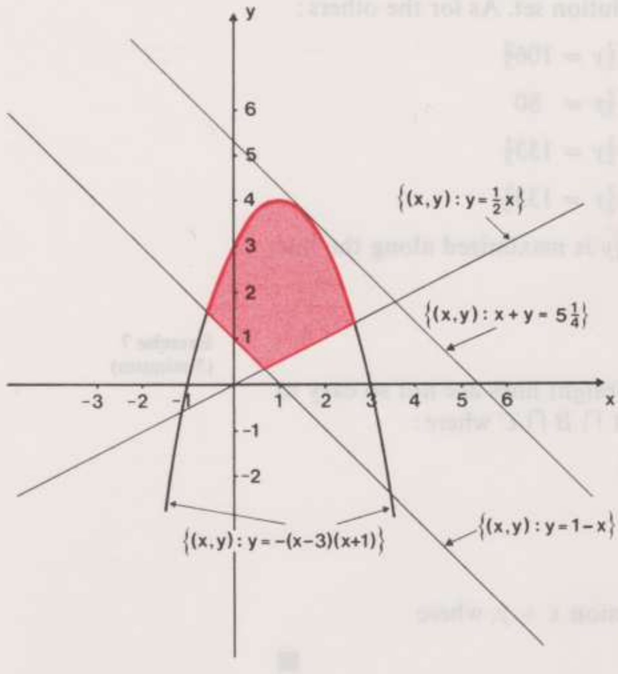
$$C = \{(x, y): y \geqslant \tfrac{1}{2}x\}$$

Estimate the maximum value of the expression $x + y$, where

$(x, y) \in A \cap B \cap C$ ■

Solution 7

Solution 7



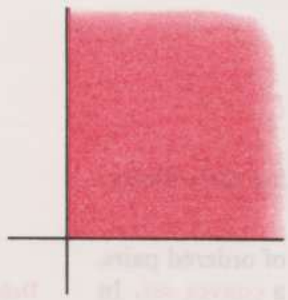
$A \cap B \cap C$ is the shaded area; the maximum value of $x + y$ where $(x, y) \in A \cap B \cap C$ is about $5\frac{1}{2}$. ■

6.3.5 Convex Sets

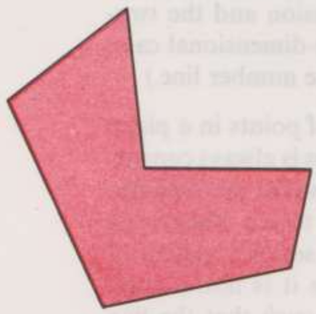
6.3.5

Discussion
* *

The regions we have been dealing with in the last two sections have all been convex, that is, they look like this:



rather than like this:



One can in fact show that the solution set of a system of linear inequalities can always be represented by a convex region. Before we do this you may like to convince yourself that this statement is true by trying to obtain an intersection of a set of half planes which is not convex.

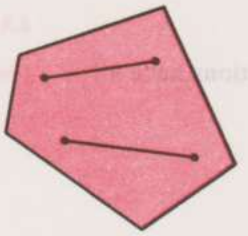
A Definition of "Convex Set"

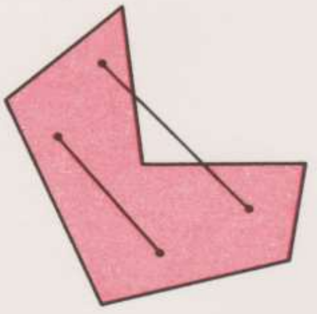
Definition
* *

If we wish to prove a statement, then we must define clearly all the terms in that statement. In this case the term about which we have been particularly vague is the term "convex set". The situation with which we are confronted often arises in mathematics. We have a concept which we can describe in vague terms, perhaps by illustration. If we wish to use the concept in a piece of mathematics, we must look for a characteristic property which can be put in mathematical terms. The property we choose here is that, for any convex region, if we take two points in the region then the straight line joining these points lies entirely in the region.

Definition 1
* *

> A region is convex, if whenever P and Q are points in the region, then every point on the line segment PQ lies in the region.





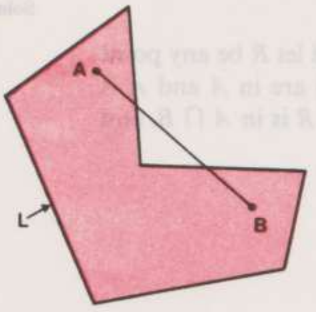
(Notice that this definition applies also to regions which are three dimensional.)

We know that a region of a plane corresponds to a set of ordered pairs. A set which corresponds to a convex region is called a convex set. In particular, if the region is bounded by straight lines, the set is called a polyhedral convex set. (The adjective "polyhedral" derives from the three-dimensional case, where the sets are represented by polyhedra. But it is used to refer both to abstract sets of higher dimension and the two-dimensional case, where we get polygons, and the one-dimensional case, where a convex set is represented by an interval on the number line.)

Definition 2

Definition 3

We can now prove our earlier assertion, that the set of points in a plane which is the solution set of a system of linear inequalities is always convex. One way to prove it is to assume the contrary, and show that this assumption leads to a contradiction. (We shall have more to say about this method of proof in *Unit 17, Logic II.*) The solution set of a system of linear inequalities will always be a polygon. Suppose it is not convex. Then we can find two points A and B in the region such that the line segment AB is not wholly within the region.



At A, every inequality is satisfied. As we move along AB we shall at some stage cross the boundary line L and one of the inequalities will cease to be satisfied. Continuing along AB, the straight line forming part of L which we have crossed will not be crossed again, and so at B the corresponding inequality will still not be satisfied, and yet B is supposed to be in the solution set. We thus arrive at a contradiction and so our original assumption, that the region is not convex, must be discarded.

Exercise 1

Exercise 1
(2 minutes)

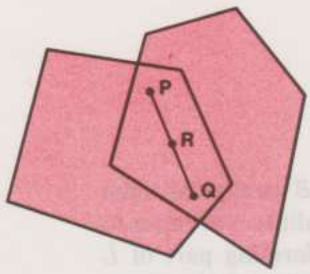
A and B are plane convex sets.

(i) Is $A \cap B$ necessarily convex?
(ii) Is $A \cup B$ necessarily convex?

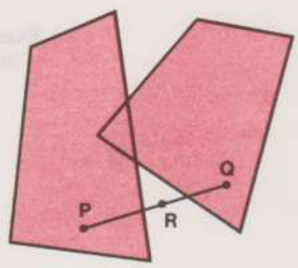
Solution 1

Solution 1

(i) YES. For let P, Q be any two points in $A \cap B$, and let R be any point on the line segment joining them. Since P and Q are in A and A is convex, R is in A. Similarly, R is in B. Therefore, R is in $A \cap B$, and so $A \cap B$ is convex.



(ii) NO. Consider the diagram below, as an illustration of a counter-example.



■

6.3.6 Summary

6.3.6

Summary

Inequalities involving one variable arise from mappings from R to R. But, since the inequality statement is about elements in the codomain, we can consider inequalities which arise from mappings from any set to R. As an example we consider mappings from $R \times R$ to R. The solution sets of these inequalities can be illustrated as two-dimensional regions, compared with the one-dimensional (number line) illustrations for inequalities involving one variable. This is to be expected — the solution sets are subsets of the domain; R can be represented by a line and $R \times R$ by a plane. If we considered inequalities which arise from mappings of $R \times R \times R$ to R, the solution sets would be represented by a three-dimensional region.

The regions which arise from systems of linear inequalities in two (or indeed more than two) variables are examples of convex regions. These are important in the "Linear Programming Problem". This is a type of problem which arises frequently in large-scale management decisions.

We illustrated the problem with an example which, whilst being unrealistic and rather trivial, does nevertheless demonstrate the important principles involved.

M100—MATHEMATICS FOUNDATION COURSE UNITS

1 Functions
2 Errors and Accuracy
3 Operations and Morphisms
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5 NO TEXT
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8 Computing I
9 Integration I
10 NO TEXT
11 Logic I—Boolean Algebra
12 Differentiation I
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